

Radial Deformations and Cavitation in Riemannian Manifolds with Applications to Membrane Shells

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Abstract This study is a geometric version of Ball's work, Philos. Trans. Roy. Soc. London Ser. A 306 (1982), no. 1496, 557-611. Radial deformations in Riemannian manifolds are singular solutions to some nonlinear equations given by constitutive functions and radial curvatures. A geodesic spherical cavity forms at the center of a geodesic ball in tension by means of given surface tractions or displacements. The existence of such solutions depends on the growth properties of the constitutive functions and the radial curvatures.

Some close relationships are shown among radial curvature, the constitutive functions, and the behavior of bifurcation of a singular solution from a trivial solution. In the incompressible case the bifurcation depends on the local properties of the radial curvature near the geodesic ball center but the bifurcation in compressible case is determined by the global properties of the radial curvatures.

A cavity forms at the center of a membrane shell of isotropic material placed in tension by means of given boundary tractions or displacements when the Riemannian manifold under question is a surface of \mathbb{R}^3 with the induced metric. In addition, cavitation at the center of ellipsoids of \mathbb{R}^n is also described if the Riemannian manifold under question is (\mathbb{R}^n, g) where $g(x)$ are symmetric, positive matrices for $x \in \mathbb{R}^n$.

Keywords radial curvature, cavitation, exponential map, membrane shell

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1 Introduction

The present paper is a geometric version of Ball's work [2].

We investigate a class of singular solutions to the problems in which a hole forms in the center of a geodesic ball in a state of tension on a Riemannian manifold. This phenomenon of hole formation is said to be *cavitation* by a terminology commonly used in the special case of an elastic fluid.

The study of cavitation in the Euclidean space was initiated by Ball in the fundamental paper [2]. The work of Ball was, in part, motivated by the work of [3] and subsequently developed by many authors (see, e.g., [7, 8, 10, 11, 12, 13, 14, 15] and the review article [5]).

Let (M, g) be a n -dimensional Riemannian manifold. Consider a body having strain energy W in which the body occupies the open subset Ω of M . In a typical deformation in which a particle $x \in M$ is displaced to $\mathbf{u}(x) \in M$ the strain energy is given by

$$E(\mathbf{u}) = \int_{\Omega} W(d\mathbf{u})dg. \quad (1.1)$$

The equilibrium equations of the body with zero body force are the Euler-Lagrange equations for the integral above. Solutions to these equilibrium equations are said to be equilibrium solutions.

Let $o \in M$ be given and let $\exp_o : M_o \rightarrow M$ be the exponential map. Radial deformations of a geodesic ball $\Omega = \{\exp_o \rho v \mid 0 \leq \rho < 1, v \in M_o, |v| = 1\}$ have the form

$$\mathbf{u}(x) = \exp_o \varphi(\rho) v \quad \text{for } x = \exp_o \rho v \in \Omega. \quad (1.2)$$

Radial curvatures play a key role in the structure of radial deformations above. To get equilibrium equations for (1.2) from (1.1), we consider the case where radial curvatures on the geodesic sphere centered at o with radius $t > 0$ are the same, denoted by $\kappa(t)$, for which we say (M, g, o) is a *model* ([4]). We also assume that $W(F)$ can be expressed as a symmetric function $\Phi(v_1, \dots, v_n)$ of the eigenvalues of $(F^T F)^{1/2}$. For the incompressible case the only kinematically admissible deformations of the form (1.2) are given by

$$\varphi(\rho) = \sigma^{-1}(\sigma(\rho) + \sigma(A)) \quad \text{for } \rho \geq 0, \quad (1.3)$$

where $A = \varphi(0) \geq 0$ and function σ is defined by

$$\sigma(t) = \int_0^t f^{n-1}(s) ds \quad \text{for } t \geq 0, \quad (1.4)$$

where f is the solution to problem

$$f''(t) + \kappa(t)f(t) = 0 \quad \text{for } t > 0; \quad f(0) = 0, \quad f'(0) = 1. \quad (1.5)$$

For the compressible case φ has to satisfy the radial equilibrium equation (Theorem 2.1)

$$[f^{n-1}(\rho)\Phi_1(\rho)]_\rho = (n-1)f^{n-2}(\rho)f' \circ \varphi(\rho)\Phi_2(\rho) \quad \text{for } x \in \Omega, \quad \rho(x) > 0, \quad (1.6)$$

where

$$\Phi_i(\rho) = \Phi_{v_i}(\varphi'(\rho), \tau(\rho), \dots, \tau(\rho)), \quad \tau(\rho) = \frac{f \circ \varphi(\rho)}{f(\rho)} \quad \text{for } \rho > 0, \quad i = 1, 2,$$

where f is the solution to problem (1.5). Equation (1.6) expresses the close relationship among radial curvature κ , constitutive function W and radial deformation φ . Cavitation is equivalent to proving existence of solutions to problem (1.6) such that $\varphi(0) > 0$.

Consider existence of equilibrium solutions in the incompressible case. The assumption on the radial curvature such that formula (1.3) makes sense is the following

$$\int_0^{\max\{1, \varphi(1)\}} s \kappa_+(s) ds \leq 1, \quad (1.7)$$

where $\kappa_+(s) = \max\{0, \kappa(s)\}$. We show (Theorem 2.3) that under assumption (1.7) if $A > 0$ then (1.3) generates an equilibrium solution if and only if

$$\frac{v^{n-1}}{(v^n - 1)^2} \hat{\Phi}'(v) \in L^1(\delta, \infty) \quad \text{for } \delta > 1, \quad (1.8)$$

where $\hat{\Phi}(v) = \Phi(v^{n-1}, v, \dots, v)$. Conditions (1.8) are the same as in [2] in the Euclidean space. Let P be the radial component of the Piola-Kirchhoff stress at $\rho = 1$ and let T be the Cauchy stress. Under assumptions (1.7) and (1.8) and with the choice $T(0) = 0$ a critical value P_{cr} of P for solutions of bifurcation is given by

$$P_{cr} = \int_1^\infty \frac{1}{v^n - 1} \hat{\Phi}'(v) dv, \quad (1.9)$$

which is again the same as in [2] in the Euclidean space.

In the compressible case the establishment of existence of cavitating equilibrium solutions is much more complicated. We use some similar assumptions on the growth properties of the constitutive function W as in [2] to analyze equilibrium solutions. The displacement boundary value problem in which $\varphi(1) = \lambda > 0$ is specified is concerned. A solution to problem (1.6) is said to be *regular* if $\varphi(0) = 0$. Since there are no explicit formulas for regular equilibrium solutions in general, we establish some estimates of regular equilibrium solutions from below and above (Theorems 4.1 and 4.2) under the radial curvature assumption (1.7). Using these estimates for regular equilibrium solutions and under the radial curvature assumptions

$$\int_0^\infty s \kappa_+(s) ds \leq 1, \quad \int_0^\infty s \kappa_-(s) ds < \infty, \quad (1.10)$$

where $\kappa_-(s) = \max\{0, -\kappa\}$, we show (Theorem 4.8) that for λ large enough there is a unique radial minimizer φ of E with $\varphi(1) = \lambda$ and $\varphi(0) > 0$, which is a stable cavitating equilibrium solution.

One of the direct applications of the analysis here is the cavitation problem of membrane shells. Let M be a surface in \mathbb{R}^3 with the induced metric g . Suppose that the middle surface of a shell is a bounded open set $\Omega \subset M$. First we show (Proposition 5.1) that the Γ -limit membrane shell, given in [6], takes the form (1.1) if all deformations of the middle are confined in M . So we assume that the membrane shells have their stored energies in the form (1.1) to study their cavitation problems. In particular, the following surfaces of revolution are concerned:

$$M = \{ (x, \psi(r)) \in \mathbb{R}^3 \mid x = (x_1, x_2) \in \mathbb{R}^2, \ r = |x| \},$$

where ψ is a C^2 function on $[0, \infty)$. Then (M, g, o) is a model where g is the induced metric of M from \mathbb{R}^3 and $o = (0, 0, \psi(0))$. The radial curvature is given by

$$\kappa(t) = \frac{\psi'(\zeta(t))\psi''(\zeta(t))}{\zeta(t)(1 + \psi'^2(\zeta(t)))^2} \quad \text{for } t \geq 0,$$

where function $\zeta(t)$ is defined by equation

$$t = \int_0^{\zeta(t)} \sqrt{1 + \psi'^2(s)} ds \quad \text{for } t \geq 0.$$

In the incompressible case a critical value of the radial component of the Piola-Kirchhoff stress P at $\rho = 1$ for solutions of bifurcation is given by (1.9) under the assumptions (1.7), (1.8) and $T(0) = 0$. In the compressible case when $\varphi(1) = \lambda$ is large enough the stable solution is cavitating ($\varphi(0) > 0$) under the assumptions (1.10), $T(0) = 0$ and the growth assumptions of W .

Consider the Riemannian manifold (\mathbb{R}^n, g) where $g = G(x)$ are symmetric and positive matrices for $x \in \mathbb{R}^n$. The radial deformation theory of Sections 2-4 describes that a hole forms in the center of an ellipsoid in a state of tension in Section 6.

2 Equilibrium Equations for Radial Deformations on a Model

We make some preparations for our problems. Let (M, g) be a n -dimensional Riemannian manifold with an orientation and let $\Omega \subset M$ be an open set. A map $\mathbf{u} : \Omega \rightarrow M$ is said to be a *deformation*. Let $\mathbf{u} : \Omega \rightarrow M$ be a deformation. We define the *deformation gradient* $d\mathbf{u}$ of \mathbf{u} as a bilinear functional on $M_{\mathbf{u}(x)} \times M_x$ by

$$d\mathbf{u}(Y, X) = \langle Y, \mathbf{u}_* X \rangle \circ \mathbf{u}(x) \quad \text{for } Y \in M_{\mathbf{u}(x)}, \quad X \in M_x, \quad x \in \Omega, \quad (2.1)$$

where $\langle \cdot, \cdot \rangle = g$ is the Riemannian metric.

Let $W : M_+^{n \times n} \rightarrow \mathbb{R}$ be a function where $M_+^{n \times n}$ is the set of real $n \times n$ matrices with positive determinant. We denote by $\text{SO}(n)$ the special orthogonal group on \mathbb{R}^n . We assume that

$$W(F) = W(QF) = W(FQ) \quad \text{for } F \in M_+^{n \times n}, \quad Q \in \text{SO}(n). \quad (2.2)$$

Let $x \in M$ be given. Let $\{e_i\}$ and $\{E_i\}$ be orthonormal bases of $(M_x, g(x))$ and $(M_{\mathbf{u}(x)}, g \circ \mathbf{u}(x))$ with the positive orientation, respectively. We define

$$W(d\mathbf{u}) = W(F), \quad (2.3)$$

where

$$F = \left(d\mathbf{u}(E_i, e_j) \right). \quad (2.4)$$

We have the following.

Lemma 2.1 *Let W satisfy (2.2). Then the definition of (2.3) is independent of the selections of $\{e_i\}$ and $\{E_i\}$.*

Proof Let $\{\hat{e}_i\}$ and $\{\hat{E}_i\}$ be the different selections. Let

$$\hat{e}_i = \sum_{j=1}^n q_{ij} e_j, \quad \hat{E}_i = \sum_{j=1}^n r_{ij} E_j.$$

Then

$$\langle \hat{E}_i, \mathbf{u}_* \hat{e}_j \rangle \circ \mathbf{u}(x) = \sum_{kl} r_{ik} \langle E_k, \mathbf{u}_* e_l \rangle q_{jl},$$

that is,

$$\left(d\mathbf{u}(\hat{E}_i, \hat{e}_j) \right) = R \left(d\mathbf{u}(E_i, e_j) \right) Q^T,$$

where $R = (r_{ij})$ and $Q = (q_{ij})$ are in $\text{SO}(n)$. Then the lemma follows from (2.2). \square

Remark 2.1 Let $M = \mathbb{R}^n$ with the Euclidean metric and $\mathbf{u} = (u_1, \dots, u_n)$. Then

$$\left\langle \frac{\partial}{\partial x_i}, \mathbf{u}_* \frac{\partial}{\partial x_j} \right\rangle = \frac{\partial u_i}{\partial x_j}.$$

In a typical deformation in which the point $x \in \Omega$ is displaced to $\mathbf{u}(x) \in M$ the energy of \mathbf{u} is given by

$$E(\mathbf{u}) = \int_{\Omega} W(d\mathbf{u}) dg, \quad (2.5)$$

where dg is the volume element of M in the metric g .

Proposition 2.1 Let $x = (x_1, \dots, x_n)$ be a local coordinate system on (M, g) and

$$g = \sum_{ij=1}^n g_{ij}(x) dx_i dx_j.$$

Let $G = (g_{ij})$. Then the equilibrium equations are the Euler-Lagrange equations for (2.5)

$$\sum_{lj} \frac{\partial}{\partial x_l} \left(\frac{\partial W(d\mathbf{u}(x))}{\partial F_{ij}} \alpha_{ip}(\mathbf{u}(x)) \alpha^{jl}(x) \sqrt{\det G(x)} \right) = 0 \quad \text{for } 1 \leq p \leq n, \quad (2.6)$$

where

$$(\alpha_{ij}) = G^{1/2}, \quad (\alpha^{ij}) = G^{-1/2}.$$

Proof Let $\mathbf{v} : M \rightarrow M$ be a deformation and let

$$e_i(x) = \sum_{j=1}^n \alpha^{ij}(x) \partial x_j \quad \text{for } 1 \leq i \leq n.$$

Then e_1, \dots, e_n form an orthonormal basis of M_x and E_1, \dots, E_n is an orthonormal basis of $M_{\mathbf{v}(x)}$, where $E_i = e_i(\mathbf{v}(x))$ for $1 \leq i \leq n$. Moreover, it follows that

$$\begin{aligned} \langle E_i, \mathbf{v}_* e_j \rangle &= \sum_{kl} \alpha^{ik}(\mathbf{v}(x)) \langle \partial x_k, \mathbf{v}_* \partial x_l \rangle \alpha^{jl}(x) \\ &= \sum_{klh} \alpha^{ik}(\mathbf{v}(x)) g_{kh}(\mathbf{v}(x)) \alpha^{jl}(x) \frac{\partial v_h}{\partial x_l} \\ &= \sum_{lh} \alpha_{ih}(\mathbf{v}(x)) \alpha^{jl}(x) \frac{\partial v_h(x)}{\partial x_l}. \end{aligned}$$

Let

$$I(\varepsilon) = E(\mathbf{u} + \varepsilon \mathbf{v}).$$

We have

$$\begin{aligned} I'(0) &= \sum_{ij} \int_{\Omega} \frac{\partial W}{\partial F_{ij}} \langle E_i, \mathbf{v}_* e_i \rangle \sqrt{\det G(x)} dx \\ &= \sum_{ijlh} \int_{\Omega} \left\{ \frac{\partial}{\partial x_l} \left(\frac{\partial W}{\partial F_{ij}} \alpha_{ih}(\mathbf{v}(x)) \alpha^{jl}(x) \sqrt{\det G(x)} v_h \right) \right. \\ &\quad \left. - \left[\frac{\partial}{\partial x_l} \left(\frac{\partial W}{\partial F_{ij}} \alpha_{ih}(\mathbf{v}(x)) \alpha^{jl}(x) \sqrt{\det G(x)} \right] v_h \right\} dx. \end{aligned}$$

Equations (2.6) follow. \square

Remark 2.2 Let $M = \mathbb{R}^n$ with the Euclidean metric. Equations (2.6) become

$$\sum_{j=1}^n \frac{\partial}{\partial x_j} \left[\frac{\partial W(d\mathbf{u})}{\partial F_{ij}} \right] = 0 \quad \text{for } 1 \leq i \leq n.$$

Remark 2.3 For a general deformation \mathbf{u} , the problem (2.6) may be very complicated.

Let

$$W(F) = \frac{1}{2} |F|^2 \quad \text{for } F \in M_+^{n \times n}.$$

Then equations (2.6) are

$$\Delta u_p + \sum_{ijkl=1}^n g^{ij}(x) \Gamma_{kl}^p(\mathbf{u}(x)) \frac{\partial u_k}{\partial x_i} \frac{\partial u_l}{\partial x_j} = 0 \quad \text{for } 1 \leq p \leq n,$$

where Δ is the Laplacian on M in the metric g and Γ_{ij}^k are the Christoffel symbols. Solutions of the above equations are called harmonic maps, see Lemma 8.1.1 in [9].

We are interested in radical deformations that are introduced below.

Let $o \in M$ be fixed and let $\exp_o : M_o \rightarrow M$ be the exponential map at the point o in the metric g . For any $x \in M$, there exists a pair (ρ, v) with $\rho \geq 0$ and such that

$$x = \exp_o \rho v \tag{2.7}$$

where $v = v(x) \in S_o$ and S_o is the unit sphere of $(M_o, g(o))$. Let $d(x, y)$ be the distance function from x to y in the metric g . Then $\rho = d(o, x)$.

Definition 2.1 A map $\mathbf{u} : M \rightarrow M$ is said to be a radical deformation with respect to $o \in M$ if there is a function $\varphi : [0, \infty) \rightarrow \mathbb{R}$ such that

$$\mathbf{u}(x) = \exp_o \varphi(\rho) v \quad \text{for } x = \exp_o \rho v \in M. \tag{2.8}$$

We shall solve the problem (2.6) when \mathbf{u} is a radical deformations under appropriate assumptions on the constitutive function W and on the geometric properties of the metric g . For this end, we need to computer $d\mathbf{u}$ first.

Let $\mathcal{X}(M)$ be all vector fields on M . Let D be the Levi-Civita connection of the metric g . Let X and Y be vector fields on M . The curvature operator is a map $\mathbf{R}_{XY} : \mathcal{X}(M) \rightarrow \mathcal{X}(M)$, given by

$$\mathbf{R}_{XY}Z = -D_X D_Y Z + D_Y D_X Z + D_{[X,Y]}Z$$

for all $Z \in \mathcal{X}(M)$, where $[\cdot, \cdot]$ is the Lie bracket product. Let $\gamma(t)$ be a geodesic with $|\dot{\gamma}(t)| = 1$ initiating from the point o . A vector field $J : [0, \infty) \rightarrow M_\gamma$ is called a Jacobi field along γ if

$$\ddot{J}(t) + \mathbf{R}_{\dot{\gamma}(t)J}\dot{\gamma} = 0 \quad \text{for } t \geq 0.$$

In addition, a Jacobi field J is said to be normal if

$$\langle J(t), \dot{\gamma}(t) \rangle = 0 \quad \text{for } t \geq 0.$$

Proposition 2.2 *Let \mathbf{u} be a radical deformation and let $\rho = \rho(x)$ be the distance function in the metric g from $x \in M$ to o . Then*

$$\mathbf{u}_* D\rho(x) = \varphi'(\rho) D\rho(\varphi(\rho)) \quad \text{for } x \in M, \quad x \neq o. \quad (2.9)$$

Let $J(t)$ be a normal Jacobi field along the geodesic $\gamma(t)$ with $J(0) = 0$. Then

$$\mathbf{u}_* J(t) = J(\varphi(t)) \quad \text{for } t \in \mathbb{R}. \quad (2.10)$$

Proof Formula (2.9) follows from expression (2.8).

Let $\gamma(t) = \exp_o tv$ where $v \in M_o$ with $|v| = 1$. Since $\langle v, \dot{J}(0) \rangle = 0$, there is a curve $\sigma : [0, 1] \rightarrow M_o$ such that

$$|\sigma(s)| = 1, \quad \sigma(0) = v, \quad \dot{\sigma}(0) = \dot{J}(0).$$

Let

$$\alpha(t, s) = \exp_o t\sigma(s) \quad \text{for } (t, s) \in [0, \infty) \times [0, 1].$$

Then

$$J(t) = \alpha_s(t, 0) = t \exp_o \dot{J}(0) \quad \text{for } t \in [0, \infty).$$

By definition, we have

$$\mathbf{u}(\alpha(t, s)) = \exp_o \varphi(t)\sigma(s) \quad \text{for } (t, s) \in [0, \infty) \times [0, 1],$$

which yields

$$\mathbf{u}_* J(t) = \varphi(t) \exp_{o*} \dot{J}(0) = J(\varphi(t)).$$

□

For any $v \in M_o$ with $|v| = 1$, there is a unique $t_0(v) > 0$ (or $t_0(v) = \infty$) such that the normal geodesic $\gamma(t) = \exp_o tv$ is the shortest on the interval $[0, t_0]$. Let

$$C(o) = \{ t_0(v)v \mid v \in M_o, |v| = 1 \}, \quad \Sigma(o) = \{ tv \mid v \in M_o, |v| = 1, 0 \leq t < t_0(v) \}.$$

The set $\exp_o C(o) \subset M$ is said to be the cut locus of o and the set $\exp_o \Sigma(o) \subset M$ is called the interior of the cut locus of o . Then

$$M = \exp_o \Sigma(o) \cup \exp_o C(o).$$

Furthermore, $\exp_o : \Sigma(o) \rightarrow \exp_o \Sigma(o)$ is a diffeomorphism and $C(o)$ is a zero measure set on M_o . Then $\exp_o C(o)$ is a zero measure set on M since it is the image of the zero measure set $C(o)$, that is, $\exp_o \Sigma(o)$ is M minus a zero measure set.

Let $\psi : M_o \rightarrow M_o$ be a linear operator. We define a map $\Psi : \exp_o \Sigma(o) \rightarrow \exp_o \Sigma(o)$ by

$$\Psi(x) = \exp_o \rho \psi v \quad \text{for } x = \exp_o \rho v \in \exp_o \Sigma(o). \quad (2.11)$$

Definition 2.2 Let $o \in M$ be fixed. The triple (M, g, o) is said to be a **model** if for every linear isometry $\psi : M_o \rightarrow M_o$, $\Psi : \exp_o \Sigma(o) \rightarrow \exp_o \Sigma(o)$ is an isometry.

Remark 2.4 A conception of models is introduced in [4]. Definition 2.2 above is weaker than that in [4]. If $\exp_o : M \rightarrow M$ is a diffeomorphism, they are the same.

For any $v \in S_o$, $\gamma(t) = \exp_o tv$ is a normal geodesic initiating from the point o . The radial curvature tensor along $\gamma(t)$ is a tensor field of order two, given by

$$\mathbf{R}(\dot{\gamma}(t), X, \dot{\gamma}(t), Y) = \langle \mathbf{R}_{\dot{\gamma}(t)X} \dot{\gamma}, Y \rangle \quad \text{for } X, Y \in M_{\gamma(t)}, \quad t \geq 0.$$

A model is characterized by its radial curvature. We have

Proposition 2.3 (M, g, o) is a model if and only if there is a function κ on $[0, \infty)$ such that

$$\mathbf{R}(\dot{\gamma}(t), X, \dot{\gamma}(t), Y) = \kappa(t) \langle X, Y \rangle, \quad X, Y \in M_{\gamma(t)}, \quad (2.12)$$

with $\langle X, \dot{\gamma}(t) \rangle = 0$ and $\langle Y, \dot{\gamma}(t) \rangle = 0$ for all $t > 0$, where $\gamma(t) = \exp_o tv \in \exp_o \Sigma(o)$, for all $v \in M_o$ with $|v| = 1$.

Proof Let (M, g, o) be a model and let $\rho > 0$ be given. Let $\gamma_i(t) = \exp_o tv_i$ with $v_i \in S_o$ for $i = 1, 2$. Let X_i be in $M_{\gamma_i(\rho)}$ with $\langle X_i, \dot{\gamma}_i(\rho) \rangle = 0$ and $|X_i| = 1$, respectively, for $i = 1, 2$. Let z_i be in S_o such that

$$z_i(\rho) = X_i,$$

where $z_i(t)$ are the parallel translations of z_i along γ_i such that $z_i(0) = z_i$, respectively, for $i = 1, 2$. Then

$$\langle z_i, v_i \rangle = 0 \quad \text{for } i = 1, 2.$$

Suppose $\psi : M_o \rightarrow M_o$ is a linear isometry such that $\psi v_1 = v_2$ and $\psi z_1 = z_2$. Then Ψ , given by (2.11), is an isometry from $\exp_o \Sigma(o)$ to $\exp_o \Sigma(o)$. Then $\Psi(\gamma_1(t)) = \gamma_2(t)$ and $\Psi_* z_1(t) = z_2(t)$. We obtain

$$\begin{aligned} \mathbf{R}(\dot{\gamma}_2(\rho), X_2, \dot{\gamma}_2(\rho), X_2)(\gamma_2(\rho)) &= \mathbf{R}(\Psi_* \dot{\gamma}_1(\rho), \Psi_* X_1, \Psi_* \dot{\gamma}_1(\rho), \Psi_* X_1)(\Psi(\gamma_1(\rho))) \\ &= \mathbf{R}(\dot{\gamma}_1(\rho), X_1, \dot{\gamma}_1(\rho), X_1)(\gamma_1(\rho)). \end{aligned}$$

Thus, the radial curvatures are the same on the geodesic sphere $\mathbf{S}(\rho)$, centered at o and with radii ρ . Formula (2.12) follows with κ being the radial curvature.

Conversely, suppose (2.12) holds. Let $\psi : M_o \rightarrow M_o$ be a linear isometry. Let $x = \exp_o \rho v \in \exp_o \Sigma(o)$ be given. Let X_i be in M_x with $|X_i| = 1$ for $i = 1, 2$. Suppose $E_i(t)$ are the parallel translations of some $v_i \in S_o$ along $\gamma(t) = \exp_o tv$ such that $E_i(\rho) = X_i$, respectively, for $i = 1, 2$. Let

$$\alpha_i(t, s) = \exp_o t(v + sv_i) \quad \text{for } t \geq 0, s \in \mathbb{R}, \quad i = 1, 2.$$

Thus, $J_i(t) = t \exp_{o*} v_i$ are Jacobi fields along $\gamma(t) = \exp_o tv$. By formula (2.12) and $J'_i(0) = v_i$, we obtain

$$J_i(t) = f(t)E_i(t) \quad \text{for } t \geq 0, i = 1, 2, \quad (2.13)$$

where f is the solution to problem (2.15) later. In addition the formulas $\Psi(\alpha_i(t, s)) = \exp_o t\psi(v + sv_i)$ imply

$$\Psi_* J_i(t) = t \exp_{o*} \psi v_i$$

are also Jacobi fields along $\Psi(\gamma(t)) = \exp_o t\psi v_i$ for $i = 1, 2$. By (2.12) again,

$$\Psi_* J_i(t) = f(t)\hat{E}_i(t), \quad (2.14)$$

where $\hat{E}_i(t)$ are parallel translation vector fields along $\Psi(\gamma(t))$, respectively, for $i = 1, 2$, such that

$$\hat{E}_i(0) = \psi v_i \quad \text{for } i = 1, 2.$$

It follows from (2.13) and (2.14) that

$$\langle \Psi_* X_1, \Psi_* X_2 \rangle = \langle \Psi_* E_1(\rho), \Psi_* E_2(\rho) \rangle = \langle \hat{E}_1(0), \hat{E}_2(0) \rangle = \langle \psi v_1, \psi v_2 \rangle = \langle v_1, v_2 \rangle = \langle X_1, X_2 \rangle.$$

Thus, $\Psi : \exp_o \Sigma(o) \rightarrow \exp_o \Sigma(o)$ is an isometry. \square

Remark 2.5 Formula (2.12) means that a model has the same radial curvature on a geodesic sphere $\mathbf{S}(t)$, centered at o with radii $t > 0$.

Let (M, g, o) be a model and let the radial curvature κ be given by (2.12). Consider problem

$$\begin{cases} f''(t) + \kappa(t)f(t) = 0, & t > 0, \\ f(0) = 0, & f'(0) = 1. \end{cases} \quad (2.15)$$

Let f be the solution to problem (2.15). Then

$$f(t) = t - \int_0^t (t-s)\kappa(s)f(s)ds, \quad (2.16)$$

which yields

$$\lim_{t \rightarrow 0+} \frac{f(t)}{t} = 1. \quad (2.17)$$

For our problems here, we need f and f' are all positive functions, for which the following is introduced. Let the radial curvature κ be given in (2.12). Let

$$\mu_{\pm}(\lambda) = \int_0^{\lambda} s\kappa_{\pm}(s)ds \quad \text{for } \lambda > 0, \quad (2.18)$$

where

$$\kappa_+(s) = \max\{\kappa(s), 0\}, \quad \kappa_-(s) = \max\{-\kappa(s), 0\} \quad \text{for } s \geq 0.$$

We have ([4])

Proposition 2.4 *If*

$$\mu_+(\lambda) \leq 1, \quad (2.19)$$

then there exists $0 < \mu_0(\lambda) \leq 1$ such that

$$\mu_0(\lambda)\rho \leq f(\rho) \leq e^{\mu_-(\lambda)}\rho \quad \text{for } \rho \in [0, \lambda], \quad (2.20)$$

and

$$\mu_0(\lambda) \leq f'(\rho) \leq e^{\mu_-(\lambda)} \quad \text{for } \rho \in [0, \lambda], \quad (2.21)$$

where f is the solution to problem (2.15).

Proof Let p_+ and p_- solve the problems

$$\begin{cases} p_+'' + \kappa_+p_+ = 0 & \text{for } \rho > 0, \\ p_+(0) = 0, & p_+'(0) = 0 \end{cases} \quad (2.22)$$

and

$$\begin{cases} p_-'' - \kappa_-p_- = 0 & \text{for } \rho > 0, \\ p_-(0) = 0, & p_-'(0) = 0, \end{cases} \quad (2.23)$$

respectively.

Let $\eta(\rho) = p_-/p_+'$ for $\rho > 0$. By (2.23) we have

$$\eta' = 1 - \kappa_- \eta \leq 1, \quad \text{and, then } \eta \leq \rho \quad \text{for } \rho \geq 0,$$

since $\eta(0) = 0$. It follows that

$$\frac{p_-''}{p_-'} = \kappa_- \eta \leq \kappa_-(\rho)\rho \quad \text{for } \rho \geq 0.$$

Integrating the above inequality over $(0, \rho)$ yields

$$1 \leq p_-'(\rho) \leq e^{\mu_-(\lambda)} \quad \text{for } \rho \in [0, \lambda], \quad (2.24)$$

which implies

$$\rho \leq p_-(\rho) \leq e^{\mu_-(\lambda)} \rho \quad \text{for } \rho \in [0, \lambda]. \quad (2.25)$$

On the other hand, from (2.22) we obtain

$$\mu_0(\lambda) \leq p_+'(\rho) = 1 - \int_0^\rho \kappa_+(s)p_+(s)ds \leq 1, \quad (2.26)$$

and then

$$\mu_0(\lambda)\rho \leq p_+(\rho) \leq \rho \quad \text{for } \rho \in [0, \lambda], \quad (2.27)$$

where $\mu_0(\lambda) = \min_{0 \leq \rho \leq \lambda} p_+'(\lambda) \leq 1$. Moreover, we claim that assumption (2.19) implies that $\mu_0(\lambda) > 0$. Otherwise, if there were a point $\rho_0 \in [0, \lambda]$ such that $p_+'(\rho_0) = 0$, then by (2.27)

$$1 = \int_0^{\rho_0} \kappa_+(s)p_+(s)ds < \mu_+(\lambda) \leq 1,$$

a contradiction.

By a comparison argument for ordinary differential equations we obtain

$$\frac{p_+'}{p_+} \leq \frac{f'}{f} \leq \frac{p_-'}{p_-} \quad \text{for } \rho \in [0, \lambda]. \quad (2.28)$$

Integrating the above inequalities over (ε, ρ) gives

$$p_+ \frac{f(\varepsilon)}{p_+(\varepsilon)} \leq f \leq \frac{f(\varepsilon)}{p_-(\varepsilon)} p_- \quad \text{for } 0 < \varepsilon \leq \rho \leq \lambda.$$

Letting $\varepsilon \rightarrow 0+$ in the above inequalities we have

$$p_+ \leq f \leq p_- \quad \text{for } \rho \in [0, \lambda], \quad (2.29)$$

since

$$\lim_{\rho \rightarrow 0+} \frac{f(\rho)}{\rho} = \lim_{\rho \rightarrow 0+} \frac{p_\pm(\rho)}{\rho} = 1.$$

Then (2.20) follows from (2.29), (2.27), and (2.25).

Finally, (2.21) follows from (2.28), (2.29), (2.24), and (2.26). \square

The following result is immediate from Proposition 2.4 which is an improvement of Lemmas 4.5 and 4.6 in [4].

Proposition 2.5 *If*

$$\mu_+(\infty) \leq 1, \quad \mu_-(\infty) < \infty, \quad (2.30)$$

then there are $0 < \mu_0 \leq \mu_1 < \infty$ such that

$$\mu_0 \leq f' \leq \mu_1 \quad \text{and} \quad \mu_0 \rho \leq f \leq \mu_1 \rho \quad \text{for} \quad \rho \in [0, \infty). \quad (2.31)$$

Let $\rho = \rho(x)$ be the distance function in the metric g from o to $x \in M$. We recollect some properties of a model from [4] in the following.

Proposition 2.6 *Let (M, g, o) be a model. Then*

(i) *Any Jacobi field $J(t)$ is in the form*

$$J(t) = f(t)E(t) \quad \text{for} \quad t > 0,$$

where $E(t)$ is the parallel translation.

(ii) *The Hessian of the distance function ρ is given by*

$$D^2\rho = \frac{f'(\rho)}{f(\rho)}(g - D\rho \otimes D\rho) \quad \text{for} \quad \rho(x) > 0, \quad x \in \Sigma(o).$$

(iii) *In the geodesic polar coordinates the metric g has the expression:*

$$g = d\rho^2 + f^2(\rho)d\theta^2 \quad \text{for} \quad \rho > 0, \quad x \in \Sigma(o). \quad (2.32)$$

Proposition 2.7 *Let (M, g, o) be a model. Let \mathbf{u} be a radical deformation given by (2.8). Then*

$$(d\mathbf{u}) = \text{diag}(\varphi'(\rho), \tau(\rho), \dots, \tau(\rho)) \quad \text{for} \quad x \in \Sigma(o), \quad (2.33)$$

where

$$\tau(\rho) = \frac{f \circ \varphi(\rho)}{f(\rho)} \quad \text{for} \quad x \in \Sigma(o). \quad (2.34)$$

Proof Let $x = \exp \rho v$ where $v \in M_o$ with $|v| = 1$. Let $\{E_i(t)\}$ be the parallel translation orthonormal basis of $M_{\gamma(t)}$ along $\gamma(t) = \exp_o tv$ with $E_1 = v$. Then

$$E_1(t) = D\rho(\gamma(t)) \quad \text{for} \quad t > 0.$$

By Proposition 2.2 and Proposition 2.6 (i), we have

$$\langle E_1(\varphi(\rho)), \mathbf{u}_* E_1(\rho) \rangle = \langle D\rho(\varphi(\rho)), \mathbf{u}_* D\rho(x) \rangle = \varphi'(\rho),$$

$$\langle E_1(\varphi(\rho)), \mathbf{u}_* E_j(\rho) \rangle = 0 \quad \text{for} \quad 2 \leq j \leq n,$$

$$\langle E_i(\varphi(\rho)), \mathbf{u}_* E_j(\rho) \rangle = \langle E_i(\varphi(\rho)), \frac{1}{f(\rho)} \mathbf{u}_* J_j(\rho) \rangle = \frac{f \circ \varphi(\rho)}{f(\rho)} \delta_{ij},$$

for $2 \leq i, j \leq n$. □

2.1 Compressible Case

Let $\mathbf{u} : M \rightarrow M$ be a differentiable map. In order to compute equilibrium equations to energy (2.5), we consider the vector bundle $\zeta = \mathbf{u}^{-1}TM$ over the base manifold (M, g) , induced by the map \mathbf{u} ,

$$\zeta = \cup_{x \in M} M_{\mathbf{u}(x)}. \quad (2.35)$$

The projection map $\pi : \zeta \rightarrow M$ is given by

$$\pi(x, Y) = x \quad \text{for } x \in M, \quad Y \in M_{\mathbf{u}(x)}.$$

In local coordinates a *section* H of ζ is in the form

$$H(x) = \sum_{i=1}^n h_i(x) \partial_{x_i}|_{\mathbf{u}(x)} \quad \text{for } x \in M, \quad (2.36)$$

where $h_i \in C^\infty(M)$ for all i . Denote by $\Gamma(\zeta)$ all sections of ζ . The connection $D : \mathcal{X}(M) \times \Gamma(\zeta) \rightarrow \Gamma(\zeta)$, induced by the metric g , is given by

$$D_X H = \sum_{i=1}^n [X(h_i) \partial_{x_i}|_{\mathbf{u}(x)} + h_i(x) (D_{\mathbf{u}_* X} \partial_{x_i}) \circ \mathbf{u}], \quad (2.37)$$

where $X \in \mathcal{X}(M)$ and $H \in \Gamma(\zeta)$. Furthermore, for $H_1, H_2 \in \zeta$ and $X \in \mathcal{X}(M)$, we have

$$X \langle H_1, H_2 \rangle = \langle D_X H_1, H_2 \rangle + \langle H_1, D_X H_2 \rangle.$$

For $H \in \Gamma(\zeta)$, we set

$$\mathbf{v}(t)(x) = \exp_{\mathbf{u}(x)} tH(x) \quad \text{for } t \geq 0, \quad x \in M, \quad (2.38)$$

where $\exp_{\mathbf{u}(x)} : M_{\mathbf{u}(x)} \rightarrow M$ is the exponential map in the metric g at $\mathbf{u}(x)$ along the vector $H(x) \in M_{\mathbf{u}(x)}$ for each $x \in M$. For each x fixed, $\mathbf{v}(t)$ is a geodesic on M initiating from $\mathbf{u}(x)$ in the metric g and

$$\mathbf{v}(0)(x) = \mathbf{u}(x), \quad \dot{\mathbf{v}}(0)(x) = H(x) \in M_{\mathbf{u}(x)} \quad \text{for } x \in M.$$

$\mathbf{v}(t)$ is said to be a *variation* of \mathbf{u} (Chapter 8 of [9]).

Let $x \in M$ be given. For $e \in M_x$, $\mathbf{v}_*(t)e$ is a vector field along the geodesic $\mathbf{v}(t)(x)$. We have

Lemma 2.2 *Let $\mathbf{u} : M \rightarrow M$ be a differentiable map and let $\mathbf{v}(t)$ be a variation of \mathbf{u} , given by (2.38) for $t \in (-\varepsilon, \varepsilon)$. Then*

$$D_{\dot{\mathbf{v}}(0)} \mathbf{v}_* e = D_e H \quad \text{for } e \in M_x, \quad x \in M. \quad (2.39)$$

Proof We do a computation in local coordinates $x = (x_1, \dots, x_n)$. Let $H \in \Gamma(\zeta)$ be given by (2.36). Let $e = \sum_{i=1}^n \alpha_i \partial_{x_i}|_x$ and

$$\mathbf{v}(t)(x) = (v_1(t, x), \dots, v_n(t, x)) \quad \text{for } (t, x) \in (-\varepsilon, \varepsilon) \times M,$$

where

$$(v_1(0, x), \dots, v_n(0, x)) = \mathbf{u}(x), \quad \dot{\mathbf{v}}(0)x = H(x) \quad \text{for } x \in M.$$

Then

$$\dot{\mathbf{v}}(t) = \sum_{i=1}^n \dot{v}_i(t) \partial_{x_i}|_{\mathbf{v}(t)}, \quad \mathbf{v}_*(t) \partial_{x_j} = \sum_{i=1}^n v_{ix_j} \partial_{x_i}|_{\mathbf{v}(t)},$$

where $\dot{v}_i(0) = h_i(x)$ and $\mathbf{v}_*(0) \partial_{x_i} = \mathbf{u}_* \partial_{x_i}$ for $1 \leq i \leq n$. We have

$$\begin{aligned} D_{\dot{\mathbf{v}}(0)} \mathbf{v}_* e &= D_{\dot{\mathbf{v}}(0)} \left[\sum_i \left(\sum_j \alpha_j v_{ix_j} \right) \partial_{x_i}|_{\mathbf{v}(t)} \right] \\ &= \sum_i \left[\sum_j \dot{\mathbf{v}}(0) (\alpha_j v_{ix_j}) \partial_{x_i}|_{\mathbf{u}(x)} + \sum_j \alpha_j v_{ix_j} \sum_k h_k (D_{\partial_{x_k}} \partial_{x_i})|_{\mathbf{u}(x)} \right] \\ &= \sum_l \left[\sum_j \alpha_j (v_{lx_j t}(0) + \sum_{ik} v_{ix_j}(0) h_k \Gamma_{ki}^l \circ \mathbf{u}) \right] \partial_{x_l}|_{\mathbf{u}(x)} \\ &= \sum_l \left[\sum_j \alpha_j (h_{lx_j} + \sum_{ik} u_{ix_j} h_k \Gamma_{ki}^l \circ \mathbf{u}) \right] \partial_{x_l}|_{\mathbf{u}(x)}. \end{aligned} \tag{2.40}$$

It follows from (2.40) that

$$\begin{aligned} D_e H &= \sum_i [e(h_i) \partial_{x_i}|_{\mathbf{u}(x)} + h_i D_{\mathbf{u}_* e} \partial_{x_i}] \\ &= \sum_{ij} \alpha_j [h_{ix_j} \partial_{x_i}|_{\mathbf{u}(x)} + h_i \sum_k u_{kx_j} (D_{\partial_{x_k}} \partial_{x_i})|_{\mathbf{u}(x)}] \\ &= D_{\dot{\mathbf{v}}(0)} \mathbf{v}_* e. \end{aligned}$$

□

Let e_1, \dots, e_n be an orthonormal basis of M_o . Consider the usual pole coordinates (ρ, θ) on the Euclidean space \mathbb{R}^n , given by

$$\begin{cases} z_1 = \rho \cos \theta_1, \\ z_2 = \rho \sin \theta_1 \cos \theta_2 \\ \dots\dots\dots \\ z_{n-1} = \rho \sin \theta_1 \cdots \sin \theta_{n-2} \cos \theta_{n-1} \\ z_n = \rho \sin \theta_1 \cdots \sin \theta_{n-2} \sin \theta_{n-1}, \end{cases} \tag{2.41}$$

where $\theta = (\theta_1, \dots, \theta_{n-1})$ and

$$0 \leq \theta_1 \leq \pi, \quad \dots, \quad 0 \leq \theta_{n-2} \leq \pi, \quad 0 \leq \theta_{n-1} \leq 2\pi.$$

Let

$$\pi(\theta) = \frac{1}{\rho} \sum_{i=1}^n z_i e_i. \quad (2.42)$$

We have

$$\lim_{\rho \rightarrow 0+} \int_{S_o} \langle X, D\rho \rangle d\theta = \lim_{\rho \rightarrow 0+} \int_{S_o} \langle X, \exp_{o*} \pi(\theta) \rangle d\theta = \int_{S_o} \langle X(o), \pi(\theta) \rangle d\theta = 0,$$

and, thus,

$$\begin{aligned} \lim_{\rho \rightarrow 0} \int_{S_o} \frac{1}{\rho} \langle X, D\rho \rangle d\theta &= \lim_{\rho \rightarrow 0+} \int_{S_o} \frac{\langle X, D\rho \rangle - \langle X(o), \pi(\theta) \rangle \langle o \rangle}{\rho} d\theta \\ &= \int_{S_o} \langle D_{\pi(\theta)} X, \pi(\theta) \rangle d\theta. \end{aligned} \quad (2.43)$$

Let the function $W : M_+^{n \times n} \rightarrow \mathbb{R}$ satisfy assumption (2.2). It is well known ([16]) that there exists a symmetric function

$$\Phi : \mathbb{R}_{++}^n \rightarrow \mathbb{R}, \quad \mathbb{R}_{++}^n = \{ c = (c_1, \dots, c_n) \in \mathbb{R}^n, \ c_i > 0 \text{ for } 1 \leq i \leq n \},$$

such that

$$W(F) = \Phi(v_1, \dots, v_n) \quad \text{for all } F \in M_+^{n \times n}, \quad (2.44)$$

where v_1, \dots, v_n denote the singular values (or principal stretches) of F (i.e., the eigenvalues of $(F^T F)^{1/2}$). It is known ([1]) that $W \in C^r(M_+^{n \times n})$ if and only if $\Phi \in C^r(\mathbb{R}_{++}^n)$ for $r = 0, 1, 2$ or ∞ . We write $\Phi_i = \frac{\partial \Phi}{\partial v_i}$, etc. If $W \in C^1(M_+^{n \times n})$ and if $F = \text{diag}(v_1, \dots, v_n)$, $v_i > 0$, then

$$\frac{\partial W}{\partial F}(F) = \text{diag}(\Phi_1, \dots, \Phi_n), \quad (2.45)$$

where $\Phi_i = \Phi_i(v_1, \dots, v_n)$ for all i . Moreover, the symmetry of the function Φ implies

$$\Phi_i(v_1, v, \dots, v) = \Phi_2(v_1, v, \dots, v) \quad \text{for } 2 \leq i \leq n. \quad (2.46)$$

Let $\mathbf{u} : \Omega \rightarrow M$ be a deformation. We say that $\mathbf{u} \in W^{1,p}(\Omega, M)$ if

$$\int_{\Omega} |F|^p dg < \infty,$$

where F is given by (2.4) and $1 \leq p < \infty$. Moreover, we define

$$\det d\mathbf{u}(x) = \det F \quad \text{for } x \in \Omega.$$

Let $\Omega = \mathbf{B}$ be the unit geodesic ball centered at o . Let $H \in \Gamma(\zeta)$ be a section. Then the differential of H in the connection D can be defined by

$$DH(x) = \left(\langle E_i, D_{e_j} H \rangle \right) \quad \text{for } x \in M,$$

where $\{e_i\}$ and $\{E_i\}$ are orthonormal bases of M_x and $M_{\mathbf{u}}$, respectively. We say that a deformation $\mathbf{u} \in W^{1,1}(\mathbf{B}, M)$ is an *equilibrium solution* if $\det d\mathbf{u} > 0$, a.e. $x \in \mathbf{B}$, $\frac{\partial W}{\partial F_{ij}} \in L^1(\mathbf{B})$ for $1 \leq i, j \leq n$, and

$$\int_{\mathbf{B}} \langle D_F W, DH \rangle dg = 0 \quad \text{for all } H \in C_0^\infty(\mathbf{B}, \Gamma(\zeta)).$$

Remark 2.6 In general the vector bundle $\Gamma(\zeta)$, given by (2.35), may depend on the deformation \mathbf{u} . Then does DH . If $M = \mathbb{R}^n$ with the Euclidean metric, then $M_{\mathbf{u}(x)} = M_x = \mathbb{R}^n$ and a section H and its differential DH are independent of the deformation.

Theorem 2.1 Let (M, g, o) be a model with $\mu_+(1) \leq 1$. Let \mathbf{u} be a radical deformation given by (2.8) such that $\mu_+(\varphi(1)) \leq 1$. Then \mathbf{u} is an equilibrium solution to problem (2.6) if and only if $\varphi \in W^{1,1}(0, 1)$, $\varphi'(\rho) > 0$ a.e. $\rho \in (0, 1]$, $f^{n-1}\Phi_1, f^{n-1}\Phi_2 \in L^1(0, 1)$, and

$$[f^{n-1}(\rho)\Phi_1(\rho)]_\rho = (n-1)f^{n-2}(\rho)f' \circ \varphi(\rho)\Phi_2(\rho) \quad \text{for } x \in \Omega, \quad \rho(x) > 0, \quad (2.47)$$

where

$$\Phi_i(\rho) = \Phi_i(\varphi', \tau, \dots, \tau) \quad \text{for } \rho > 0, \quad i = 1, 2,$$

and τ is given by (2.34).

Proof Let H be a section of $\Gamma(\zeta)$ with a compact support such that $\text{supp } H \subset \mathbf{B}$ and let $\mathbf{v}(t) = \exp_{\mathbf{u}(x)} tH(x)$ be a variation of \mathbf{u} . Let $x = \exp_o \rho v \in \Sigma(o)$ be given, where $v \in M_o$ with $|v| = 1$. Let $\{E_i\}$ be an orthonormal basis of M_o with $E_1 = v$. We transport $\{E_i\}$ along the geodesic $\gamma(t) = \exp_o tv$ parallelly to obtain the orthonormal bases $\{E_i(t)\}$ of $M_{\gamma(t)}$ for $t \geq 0$. Next, we transport parallelly the orthonormal basis $\{E_i(\varphi(\rho))\}$ of $M_{\mathbf{u}(x)}$ along the geodesic $\mathbf{v}(t) = \exp_{\mathbf{u}(x)} tH$ to have the orthonormal bases $\{\hat{E}_i(t)\}$ of $M_{\mathbf{v}(t)}$ for which

$$\hat{E}_i(0) = E_i(\varphi(\rho)) \quad \text{for } 1 \leq i \leq n. \quad (2.48)$$

In particular,

$$\hat{E}_1(0) = E_1(\varphi(\rho)) = D\rho(\varphi(\rho)). \quad (2.49)$$

By Propositions 2.2 and Proposition 2.6 (i), we have

$$\mathbf{u}_* E_1(\rho) = \varphi'(\rho) D\rho(\varphi(\rho)), \quad \mathbf{u}_* E_i(\rho) = \tau(\rho) E_i(\varphi(\rho)) \quad \text{for } 2 \leq i \leq n. \quad (2.50)$$

Let

$$F(t) = \left(\langle \hat{E}_i(t), \mathbf{v}_*(t) E_j(\rho) \rangle \right).$$

By Proposition 2.7,

$$F(0) = \text{diag}(\varphi', \tau, \dots, \tau),$$

since $\mathbf{v}_*(0) = \mathbf{u}_*$. We obtain, by Lemma 2.2, (2.48), (2.45), (2.46) and (2.49),

$$\begin{aligned}
\frac{d}{dt}W(d\mathbf{v})\Big|_{t=0} &= \sum_{ij=1}^n \frac{\partial W}{\partial F_{ij}}(F(0)) \frac{\partial \langle \hat{E}_i(t), \mathbf{v}_*(t) E_j(\rho) \rangle}{\partial t} \Big|_{t=0} \\
&= \sum_{ij=1}^n \frac{\partial W}{\partial F_{ij}}(F(0)) \langle \hat{E}_i(0), D_{E_j(\rho)} H \rangle \circ \mathbf{u}(x) \\
&= \Phi_1(\rho) \langle D\rho, D_{D\rho} H \rangle \circ \mathbf{u}(x) + \sum_{i=2}^n \Phi_i \langle E_i(\varphi(\rho)), D_{E_i(\rho)} H \rangle \circ \mathbf{u}(x) \\
&= \frac{\partial}{\partial \rho} [\Phi_1(\rho) \langle H, D\rho \rangle] - \left[\frac{\partial}{\partial \rho} \Phi_1(\rho) \right] \langle H, D\rho \rangle \\
&\quad + \Phi_2(\rho) \sum_{i=2}^n \langle E_i(\varphi(\rho)), D_{E_i(\rho)} H \rangle \circ \mathbf{u}(x). \tag{2.51}
\end{aligned}$$

Next, let

$$H = h_0(x) D\rho|_{\mathbf{u}(x)} + \tilde{H}, \tag{2.52}$$

where

$$\tilde{H} = \sum_{j=1}^{n-1} h_j(x) \partial \theta_j \Big|_{\mathbf{u}(x)}$$

and (ρ, θ) are the geodesic polar coordinates on M initiating from o in the metric g . It follows from (2.52) that $\langle H, D\rho \rangle = h_0(x)$ for $x \in M$, i.e., $\langle H, D\rho \rangle$ is a function on M , which is independent of deformation \mathbf{u} .

Using (2.50) and Proposition 2.6 (ii), we obtain

$$\begin{aligned}
\langle E_i(\varphi(\rho)), D_{E_i(\rho)} H \rangle &= \langle E_i(\varphi(\rho)), E_i(\rho)(h_0) D\rho|_{\mathbf{u}(x)} + h_0(x) D_{\mathbf{u}_* E_i(\rho)} D\rho + D_{E_i(\rho)} \tilde{H} \rangle \\
&= \langle H, D\rho \rangle \tau(\rho) D^2 \rho(E_i(\varphi(\rho)), E_i(\varphi(\rho))) + \langle E_i(\varphi(\rho)), \tilde{D}_{E_i(\rho)} \tilde{H} \rangle \\
&= \frac{\langle H, D\rho \rangle f' \circ \varphi(\rho)}{f(\rho)} + \langle E_i(\varphi(\rho)), \tilde{D}_{E_i(\rho)} \tilde{H} \rangle, \tag{2.53}
\end{aligned}$$

for $2 \leq i \leq n$, where \tilde{D} is the induced connection of $\mathbf{S}(\varphi(\rho))$ from the metric g and where $\mathbf{S}(\varphi(\rho))$ denotes the geodesic sphere with radii $\varphi(\rho)$ centered at o .

Let functions p_j on $\mathbf{S}(\varphi(\rho))$ be defined by

$$p_j(\mathbf{u}(x)) = h_j(x) \quad \text{for} \quad \mathbf{u}(x) \in \mathbf{S}(\varphi(\rho)), \quad 1 \leq j \leq n-1.$$

Using (2.37), we obtain

$$\begin{aligned}
\tilde{D}_{E_i(\rho)} \tilde{H} \Big|_{\mathbf{u}(x)} &= \sum_{j=1}^{n-1} [E_i(h_j)(x) \partial \theta_j \Big|_{\mathbf{u}(x)} + h_j(x) \tilde{D}_{\mathbf{u}_* E_i(\rho)} \partial \theta_j \circ \mathbf{u}(x)] \\
&= \tau \sum_{j=1}^n [E_i(p_j)(\mathbf{u}) \partial \theta_j \Big|_{\mathbf{u}} + p_j(\mathbf{u}) \tilde{D}_{E_j(\varphi(\rho))} \partial \theta_j \circ \mathbf{u}].
\end{aligned}$$

Thus,

$$\sum_{i=2}^n \langle E_i(\varphi(\rho)), D_{E_i(\rho)} \tilde{H} \rangle \circ \mathbf{u}(x) = \tau \operatorname{div} \tilde{H}_0 \circ \mathbf{u}(x), \quad (2.54)$$

where div is the divergence of the induced metric on $\mathbf{S}(\varphi(\rho))$ from the metric g and \tilde{H}_0 is a vector field on $\mathbf{S}(\varphi(\rho))$, given by

$$\tilde{H}_0 = \sum_{j=1}^{n-1} p_i(\mathbf{u}) \partial \theta_j \Big|_{\mathbf{u}(x)} \quad \text{for } \mathbf{u}(x) \in \mathbf{S}(\varphi(\rho)). \quad (2.55)$$

By inserting (2.53) and (2.54) into (2.51), we have

$$\begin{aligned} \frac{d}{dt} W(d\mathbf{v}) \Big|_{t=0} &= \frac{\partial}{\partial \rho} [\Phi_1(\rho) \langle H, D\rho \rangle] + [(n-1) \frac{f' \circ \varphi(\rho)}{f(\rho)} \Phi_2(\rho) - \frac{\partial}{\partial \rho} \Phi_1(\rho)] \langle H, D\rho \rangle \\ &\quad + \tau(\rho) \Phi_2(\rho) (\operatorname{div} \tilde{H}_0) \circ \mathbf{u}(x). \end{aligned} \quad (2.56)$$

Using (2.32) and (2.56), we obtain

$$\begin{aligned} \int_{\mathbf{B}} \langle D_F W, DH \rangle dg &= \int_{\mathbf{B}} \frac{d}{dt} W(d\mathbf{v}) \Big|_{t=0} dg = \int_0^1 \int_{S_o} \frac{d}{dt} W(d\mathbf{v}) \Big|_{t=0} f^{n-1}(\rho) d\rho d\theta \\ &= \int_0^1 \left\{ (n-1) f^{n-2}(\rho) f' \circ \varphi(\rho) \Phi_2(\rho) - [f^{n-1}(\rho) \Phi_1(\rho)]_\rho \right\} d\rho \int_{S_o} \langle H, D\rho \rangle d\theta \\ &\quad + \int_0^1 \tau^{2-n} \Phi_2(\rho) d\rho \int_{\mathbf{S}(\varphi(\rho))} \operatorname{div} \tilde{H}_0 d\tilde{g} \\ &\quad + \int_{S_o} (f^{n-1} \Phi_1 \langle H, D\rho \rangle d\theta \Big|_{\rho=1} - \lim_{\varepsilon \rightarrow 0} f^{n-1}(\varepsilon) \Phi_1(\varepsilon) \int_{S_o} \langle H, D\rho \rangle d\theta), \end{aligned} \quad (2.57)$$

where \tilde{g} is the induced metric on $\mathbf{S}(\varphi(\rho))$ from the metric g and $S_o \subset M_o$ is the unite sphere of M_o .

Since $(\mathbf{S}(\varphi(\rho)), \tilde{g})$ is a compact manifold without a boundary, the second integral in the right hand side of (2.57) is zero.

Let radical deformation \mathbf{u} be an equilibrium solution. We take $H \in \Gamma(\zeta)$ with $\operatorname{supp} H \subset \subset \mathbf{B} \setminus \{o\}$. Then the last two integrals in the right hand (2.57) are zero. Thus, equation (2.47) follows from (2.57).

Conversely, suppose that $f^{n-1} \Phi_1, f^{n-1} \Phi_2 \in L^1(0, 1)$ and that equation (2.47) is true. It follows from equation (2.47) and the relation (2.57) that for any $H \in C_0^\infty(\mathbf{B}, \Gamma(\zeta))$

$$\int_{\mathbf{B}} \langle D_F W, DH \rangle dg = - \lim_{\varepsilon \rightarrow 0} f^n(\varepsilon) \Phi_1(\varepsilon) \frac{\varepsilon}{f(\varepsilon)} \int_{S_o} \frac{\langle H, D\rho \rangle}{\varepsilon} d\theta. \quad (2.58)$$

Since $f^{n-1} \Phi_1, f^{n-1} \Phi_2 \in L^1(\mathbf{B})$, it follows from (2.21) and (2.47) that

$$(f^n \Phi_1)_\rho = f' f^{n-1} \Phi_1 + (n-1) f' \circ \varphi f^{n-1} \Phi_2 \in L^1(0, 1).$$

Thus, $\lim_{\rho \rightarrow 0+} f^n \Phi_1 = 0$. By (2.58) and (2.43), \mathbf{u} is an equilibrium solution. \square

Remark 2.7 *The above theorem is Theorem 4.2 in [2] for the isotropic materials if $M = \mathbb{R}^n$ with the Euclidean metric. In that case $f(\rho) = \rho$ and*

$$(d\mathbf{u}) = \left(\varphi'(\rho), \frac{\varphi(\rho)}{\rho}, \dots, \frac{\varphi(\rho)}{\rho} \right).$$

Equation (2.47) becomes

$$[\rho^{n-1}\Phi_1(\rho)]_\rho = (n-1)\rho^{n-2}\Phi_2(\rho).$$

Throughout this paper a function $\varphi \in W^{1,1}(0,1)$ is said to *an equilibrium solution* to problem (2.47) if φ satisfies equation (2.47) and $\varphi'(\rho) > 0$ for $\rho \in (0,1]$, and is such that $f^{n-1}\Phi_1, f^{n-1}\Phi_2 \in L^1(0,1)$. By a similar argument in [2], we have

Theorem 2.2 *Let*

$$\Phi_{11}(v_1, v_2, \dots, v_2) > 0 \quad \text{for } v_1 > 0, v_2 > 0.$$

If $\varphi \in W^{1,1}(0,1)$ is an equilibrium solution to problem (2.47), then $\varphi \in C^1(0,1]$.

2.2 Incompressible Case

A map $\mathbf{u} : M \rightarrow M$ is said to be *incompressible* if, for any element of volume ω of $M_{\mathbf{u}(x)}$, $\mathbf{u}^*\omega$ is an element of volume of M_x for $x \in M$. Let (M, g, o) be a model and let \mathbf{u} be a radical deformation given by (2.8). Let $x = \exp_o \rho v \in \Sigma(o)$ with $v \in M_o$ and $|v| = 1$. Let $\{e_i\}$ be an orthonormal basis of M_x with the positive orientation such that $e_1 = D\rho(x)$. Let ω be a volume element of $M_{\mathbf{u}(x)}$. By Proposition 2.6 (i), we have

$$\mathbf{u}^*\omega(e_1, e_2, \dots, e_n) = \omega(\mathbf{u}_*e_1, \mathbf{u}_*e_2, \dots, \mathbf{u}_*e_n) = \varphi'(\rho)\tau^{n-1}(\rho),$$

where τ is given by (2.34). Thus, \mathbf{u} is incompressible if and only if

$$\varphi'(\rho)f^{n-1} \circ \varphi(\rho) = f^{n-1}(\rho) \quad \text{for } \rho > 0.$$

We define

$$\sigma(t) = \int_0^t f^{n-1}(s)ds \quad \text{for } t \geq 0. \quad (2.59)$$

Then only possible such deformations satisfy

$$\sigma(\varphi(\rho)) = \int_0^\rho f^{n-1}(s)ds + \eta, \quad (2.60)$$

where η is a constant. To get φ from the above equation, we need some assumptions on the radical curvature.

Let conditions (2.30) hold true. It follows from (2.60) that for an incompressible deformation

$$\varphi(\rho) = \sigma^{-1}\left(\sigma(\rho) + \sigma(A)\right), \quad (2.61)$$

where $A = \varphi(0)$.

Lemma 2.3 *Let (M, g, o) be a model with $\mu_+(1) \leq 1$. Then the radical deformations with $\mu_+(\varphi(1)) \leq 1$, which are incompressible, belong to $W^{1,p}(B, M)$ for $1 \leq p < n$.*

Proof We need to prove

$$\int_0^1 \left[\varphi'^2(\rho) + (n-1)\tau^2(\rho) \right]^{p/2} f^{n-1}(\rho) d\rho < \infty \quad \text{for } 1 \leq p < n. \quad (2.62)$$

If $\varphi(0) = 0$, then $\varphi(\rho) = \rho$. The above estimate is trivial. Let us assume that $\varphi(0) = A > 0$. Then estimate (2.62) follows from (2.17). \square

Let $\mathbf{u} : M \rightarrow M$ be a deformation. The determinant of $d\mathbf{u}$ is given by

$$\det d\mathbf{u}(x) = \det \left(\langle E_i, \mathbf{u}_* e_i \rangle \right) \quad \text{for } x \in M,$$

where $\{e_i\}$ and $\{E_i\}$ are orthonormal bases of M_x and $M_{\mathbf{u}(x)}$, respectively.

Let W be a constitutive function satisfying (2.2) and let \mathbf{B} be the unit geodesic ball centered at o . The equilibrium equations for incompressible radial deformations are the Euler-Lagrange equations for the functional

$$I(\mathbf{u}) = \int_{\mathbf{B}} \{W(d\mathbf{u}) - p(x)[\det d\mathbf{u} - 1]\} dg,$$

where the pressure $p(x)$ is a Lagrange multiplier corresponding to the constraint of incompressibility. Then a deformation $\mathbf{u} : \mathbf{B} \rightarrow M$ is said to be an equilibrium solution ([2]) with corresponding measurable pressure $p(x)$ if $\det \mathbf{u} = 1$ a. e. in \mathbf{B} ,

$$\partial W(d\mathbf{u}) / \partial F_{ij} - p(x)(\text{adj } d\mathbf{u})_{ij} \in L^1(\mathbf{B}) \quad \text{for } 1 \leq i, j \leq n,$$

and

$$\int_{\mathbf{B}} \langle D_F W - p D_F \det d\mathbf{u}, DH \rangle dg = 0 \quad \text{for } H \in C_0^\infty(\mathbf{B}, \Gamma(\zeta)).$$

Let

$$\hat{\Phi}(v) = \Phi(v^{1-n}, v, \dots, v) \quad \text{for } v > 0.$$

The following theorem is Theorem 4.3 in [2] if $M = \mathbb{R}^n$ is the Euclidean space.

Theorem 2.3 *Let (M, g, o) be a model with $\mu_+(1) \leq 1$. The radical deformation (2.61) with $A > 0$ and $\mu_+(\varphi(1)) \leq 1$ is an equilibrium solution if and only if*

$$\frac{\tau^{n-1}}{(\tau^n - 1)^2} \hat{\Phi}'(\tau) \in L^1(\delta, \infty) \quad \text{for } \delta > 1. \quad (2.63)$$

In this case the corresponding pressure is given by

$$p = \int_\rho^1 \frac{f' \circ \varphi(\rho)}{f \circ \varphi(\rho)} \tau^{2-n}(\rho) \hat{\Phi}'(\tau(\rho)) d\rho + \tau^{1-n}(\rho) \Phi_1(\rho) + c, \quad (2.64)$$

where $\Phi_1(\rho) = \Phi_1(\tau^{1-n}, \tau, \dots, \tau)$, $\tau = \tau(\rho)$ is given by (2.34), and c is a constant.

Proof Let p be defined by (2.64). It is easy to check that $X = \Phi_1 - p\tau^{n-1}$ satisfies the equation

$$X_\rho + (n-1)\left(\frac{f'}{f} - \frac{\varphi' f' \circ \varphi}{f \circ \varphi}\right)X - \frac{f' \circ \varphi}{f} \hat{\Phi}'(\tau) = 0 \quad \text{for } \rho > 0, \quad (2.65)$$

where p is given by (2.64).

Let H be a section of $\Gamma(\zeta)$ with a compact support on \mathbf{B} and let $\mathbf{v}(t)$ be a variation of \mathbf{u} , given by (2.38). Let $x = \exp_o \rho v \in \Sigma(o)$ be given, where $v \in M_o$ with $|v| = 1$. Let $\{E_i\}$ be an orthonormal basis of M_o with $E_1 = v$. We transport $\{E_i\}$ along the geodesic $\gamma(t) = \exp_o tv$ parallelled to obtain the orthonormal bases $\{E_i(t)\}$ of $M_{\gamma(t)}$ for $t \geq 0$. Next, we transport parallelly the orthonormal basis $\{E_i(\varphi(\rho))\}$ of $M_{\mathbf{u}(x)}$ along the geodesic $\mathbf{v}(t) = \exp_{\mathbf{u}(x)} tH$ to have the orthonormal bases $\{\hat{E}_i(t)\}$ of $M_{\mathbf{v}(t)}$ for $t \geq 0$ such that the relations (2.48), (2.49) and (2.50) hold.

Denote

$$P_i(t) = \left(\langle \hat{E}_i(t), \mathbf{v}_*(t)E_1 \rangle, \dots, \langle \hat{E}_i(t), \mathbf{v}_*(t)E_n \rangle \right)^T \quad \text{for } 1 \leq i \leq n.$$

By (2.48), (2.39), (2.50), (2.52), (2.53) and (2.54), we have

$$\begin{aligned} \left. \frac{\partial P_i}{\partial t} \right|_{t=0} &= \left(\langle \hat{E}_i(0), D_{\mathbf{v}(0)} \mathbf{v}_* E_1 \rangle, \dots, \langle \hat{E}_i(0), D_{\mathbf{v}(0)} \mathbf{v}_* E_n \rangle \right)^T \\ &= \left(\langle E_i \circ \varphi(\rho), D_{E_1} H \rangle, \dots, \langle E_i \circ \varphi(\rho), D_{E_n} H \rangle \right)^T \quad \text{for } 1 \leq i \leq n \end{aligned}$$

and

$$\begin{aligned} \left. \frac{\partial \det d\mathbf{v}}{\partial t} \right|_{t=0} &= \sum_{k=1}^n \det \left(P_1(0), \dots, \dot{P}_k(0), \dots, P_n(0) \right) \\ &= \tau^{n-1} \langle D\rho, D_{D\rho} H \rangle(\mathbf{u}(x)) + \tau^{n-2} \varphi' \sum_{i=2}^n \langle E_i \circ \varphi, D_{E_i(\rho)} H \rangle \\ &= \tau^{n-1} \langle D\rho, D_{D\rho} H \rangle(\mathbf{u}(x)) + (n-1) \frac{f' \circ \varphi}{f \circ \varphi} \langle H, D\rho \rangle \\ &\quad + \tilde{\text{div}} \tilde{H}_0 \Big|_{\mathbf{u}(x)}, \end{aligned} \quad (2.66)$$

where \tilde{H}_0 is a vector field on $\mathbf{S}(\varphi)$, given by (2.55). From (2.56), (2.66) and (2.65), a similar argument as in the proof of Theorem 2.1 yields

$$\begin{aligned} \left. \frac{\partial I(\mathbf{v})}{\partial t} \right|_{t=0} &= \int_{S_o} f^{n-1} (\Phi_1 - p\tau^{n-1}) \langle H, D\rho \rangle d\theta \Big|_0^1 \\ &\quad - \int_0^1 f^{n-1} \left\{ X_\rho + (n-1) \left(\frac{f'}{f} - \frac{\varphi' f' \circ \varphi}{f \circ \varphi} \right) X - \frac{f' \circ \varphi}{f} \hat{\Phi}'(\tau) \right\} \int_{S_o} \langle H, D\rho \rangle d\rho d\theta \\ &= \int_{\mathbf{S}} [\Phi_1(1) - p(1)\tau^{n-1}(1)] \langle H, D\rho \rangle d\mathbf{S} \\ &\quad - \lim_{\varepsilon \rightarrow 0} f^{n-1}(\varepsilon) (\Phi_1 - p\tau^{n-1}) \int_{S_o} \langle H, D\rho \rangle d\theta, \end{aligned} \quad (2.67)$$

where \mathbf{S} is the unit geodesic sphere centered at o and p is given by (2.64).

Let a radical deformation \mathbf{u} be an equilibrium solution where φ is given by (2.61) with $\varphi(0) = A > 0$. The hypothesis that $\partial W(d\mathbf{u})/\partial F_{ij} - p(x)(\text{adj } d\mathbf{u})_{ij} \in L^1(\mathbf{B})$ for $1 \leq i, j \leq n$ imply that

$$f^{n-1}(\Phi_1 - p\tau^{n-1}), \quad f^{n-1}(\Phi_2 - p\tau^{n-2}\varphi') \in L^1(0, 1), \quad (2.68)$$

and hence

$$f^{n-1}[\Phi_2 - \tau^{-n}\Phi_1] \in L^1(0, 1). \quad (2.69)$$

Since $f(0) = 0$ and $f'(0) = 1$, we take $1 > \rho_0 > 0$ such that

$$\frac{f^n(A)}{f^n(\rho)} > \sup_{0 \leq \rho \leq \rho_0} \frac{f' \circ \varphi(\rho)}{f'(\rho)} \quad \text{for } 0 \leq \rho \leq \rho_0.$$

We have

$$\begin{aligned} (n-1) \int_0^1 f^{n-1}[\Phi_2 - \tau^{-n}\Phi_1] d\rho &= \int_0^1 f^{n-1} \hat{\Phi}'(\tau) d\rho \\ &= \int_{\tau(\rho_0)}^\infty \frac{f^n \tau^{n-1}}{f' \tau^n - f' \circ \varphi} \hat{\Phi}'(\tau) d\tau + \int_{\rho_0}^1 f^{n-1} \hat{\Phi}'(\tau) d\rho. \end{aligned} \quad (2.70)$$

Since $f^n = f^n \circ \varphi \tau^{-n}$, it follows from (2.70) that the relation (2.69) holds if and only if the relation (2.63) is true.

Conversely, let assumption (2.63) hold and p be given by (2.64). We prove that \mathbf{u} is an equilibrium solution. Similar arguments as in the proof of Theorem 4.3 in [2] show that the relations (2.68) hold, that is,

$$\partial W(d\mathbf{u})/\partial F_{ij} - p(x)(\text{adj } d\mathbf{u})_{ij} \in L^1(\mathbf{B}) \quad \text{for } 1 \leq i, j \leq n.$$

Next, using (2.68) and (2.65), we deduce

$$\lim_{\varepsilon \rightarrow 0} f^n(\varepsilon)(\Phi_1 - p\tau^{n-1}) = 0.$$

Thus, by (2.67) and (2.43), we obtain

$$\left. \frac{\partial I(\mathbf{v})}{\partial t} \right|_{t=0} = 0.$$

□

3 Cavitation in the Incompressible Case

Let the radial curvature κ satisfy

$$\mu_+(\delta_0) \leq 1 \quad \text{for some } \delta_0 > 1. \quad (3.1)$$

Then estimates (2.20) and (2.21) hold.

We need the following.

Lemma 3.1 (i) Let $1 \geq \rho_1 > 0$ be given such that

$$\kappa(\rho)f^2(\rho) + nf'^2(\rho) > 0 \quad \text{for } 0 \leq \rho \leq \rho_1. \quad (3.2)$$

We fix $0 < \rho_0 \leq \rho_1$ such that $\sigma(\rho_0) < \rho_1$. Then, for all $0 < A \leq b$, we have

$$\tau'(\rho) < 0 \quad \text{for all } 0 < \rho \leq \rho_0,$$

where

$$b = \min\{\sigma^{-1}(\rho_1 - \sigma(\rho_0)), \sigma^{-1}(\sigma(\delta_0) - \sigma(1))\}. \quad (3.3)$$

(ii) For $0 < A \leq b$ and $\tau \in [\tau(\rho_0), \infty)$ given, we solve $f \circ \varphi = \tau f$ and $\sigma(\varphi) = \sigma(\rho) + \sigma(A)$ together to have $\varphi = \varphi(A, \tau)$ and $\rho = \rho(A, \tau)$. Then

$$\varphi_A = \frac{\tau f' f^{n-1}(A)}{f^{n-1}(f' \tau^n - f' \circ \varphi)}, \quad \rho_A = \frac{f' \circ \varphi f^{n-1}(A)}{f^{n-1}(f' \tau^n - f' \circ \varphi)}, \quad (3.4)$$

for $0 < A \leq b$ and $\tau \in [\tau(\rho_0), \infty)$. Moreover, there are $c_1 \geq c_0 > 0$ such that

$$\frac{c_0 A}{(\tau^n - 1)^{1/n}} \leq \rho \leq \frac{c_1 A}{(\tau^n - 1)^{1/n}}, \quad (3.5)$$

$$\frac{c_0 A \tau}{(\tau^n - 1)^{1/n}} \leq \varphi \leq \frac{c_1 A \tau}{(\tau^n - 1)^{1/n}}, \quad (3.6)$$

$$c_0 \frac{A(\tau - 1)}{(\tau^n - 1)^{1/n}} \leq \varphi - \rho \leq c_1 \frac{A(\tau - 1)}{(\tau^n - 1)^{1/n}}, \quad (3.7)$$

for all $(A, \tau) \in (0, b] \times [\tau(\rho_0), \infty)$.

(iii) For $\tau > 1$ given,

$$\lim_{A \rightarrow 0+} \frac{A}{\rho} = (\tau^n - 1)^{1/n}, \quad \lim_{A \rightarrow 0+} \frac{A}{\varphi} = \tau^{-1}(\tau^n - 1)^{1/n}. \quad (3.8)$$

Proof (i) Since

$$\tau' = \frac{f' \circ \varphi - f' \tau^n}{f \tau^{n-1}} \quad \text{for } 0 < \rho \leq 1,$$

$\tau' < 0$ if and only if

$$\frac{f'}{f^n} > \frac{f' \circ \varphi}{f^n \circ \varphi}.$$

On the other hand, condition (3.2) implies

$$\left(\frac{f'}{f^n}\right)' = -\frac{\kappa f^2 + n f'^2}{f^{n-1}} < 0 \quad \text{for } 0 < \rho \leq \rho_1.$$

Thus, f'/f^n is strictly decreasing for $\rho \in (0, \rho_1]$. For $0 < \rho \leq \rho_0$ and $0 < A \leq \sigma^{-1}(\rho_1 - \sigma(\rho_0))$, we have

$$\rho < \varphi(\rho) \leq \rho_1,$$

which yields $f'/f^n > f' \circ \varphi / f^n \circ \varphi$.

(ii) We differentiate $f \circ \varphi = \tau f$ and $\sigma(\varphi) = \sigma(\rho) + \sigma(A)$, respectively, with respect to the variable A , and obtain (3.4).

It follows from (2.61) that

$$\int_{\rho}^{\varphi} f^{n-1} ds = \int_0^A f^{n-1} ds.$$

Estimates (2.20) and (2.21) yield

$$c_0(\varphi^n - \rho^n) \leq A^n \leq c_1(\varphi^n - \rho^n), \quad (3.9)$$

for some $c_1 \geq c_0 > 0$. Since

$$\tau^n - 1 = \frac{1}{f^n} [f^n \circ \varphi - f^n] = \frac{n}{f^n} \int_{\rho}^{\varphi} f^{n-1} f' ds,$$

using (2.20) and (2.21), we have

$$\frac{c_0}{\rho^n} (\varphi^n - \rho^n) \leq \tau^n - 1 \leq \frac{c_1}{\rho^n} (\varphi^n - \rho^n), \quad (3.10)$$

for some $c_1 \geq c_0 > 0$. Thus, (3.5) follow from (3.9) and (3.10). Similar arguments yield (3.6) and (3.7).

(iii) By (3.5), for $\tau > 1$ given,

$$\frac{1}{c_1} (\tau^n - 1)^{1/n} \leq A/\rho \leq \frac{1}{c_0} (\tau^n - 1)^{1/n}$$

for $A > 0$ small. Let $a = \lim_{A \rightarrow 0+} A/\rho$. Using (3.5) and (3.4), we have

$$a = \lim_{A \rightarrow 0+} \frac{f(A)}{f(\rho)} = \lim_{A \rightarrow 0+} \frac{f'(A)}{f'(\rho)\rho_A} = \frac{\tau^n - 1}{a^{n-1}},$$

which yields $a = (\tau^n - 1)^{1/n}$. A similar computation gives the second formula. \square

Lemma 3.2 *Let $\rho_0 \in (0, 1]$ be given in Lemma 3.1 and let $\delta > 1$ be given. Suppose $b > 0$ is given such that $1 < \tau(\rho_0) \leq \delta$ for all $0 < A \leq b$. Let*

$$p(A, \tau) = \frac{f' \circ \varphi(\tau - 1)}{f' \tau^n - f' \circ \varphi} \quad \text{for } (A, \tau) \in (0, b] \times [\tau(\rho_0), \delta]. \quad (3.11)$$

Then

$$\min\{a_0, \frac{1}{n\delta}\} \leq p(A, \tau) \leq \max\{a_1, \frac{1}{n}\} \quad \text{for } (A, \tau) \in (0, b] \times [\tau(\rho_0), \delta], \quad (3.12)$$

where

$$a_0 = \inf_{0 \leq \rho \leq \rho_0} \frac{f'^2}{\kappa f^2 + n f'^2}, \quad a_1 = \sup_{0 \leq \rho \leq \rho_0} \frac{f'^2}{\kappa f^2 + n f'^2}.$$

Proof Let

$$a = \inf_{0 < A \leq b, \tau(\rho_0) \leq \tau \leq \delta} p(A, \tau).$$

Suppose $A_k \rightarrow 0+$ and $\tau_k \in [\tau(\rho_0), \delta]$ such that

$$p(A_k, \tau_k) \rightarrow a \quad \text{as } k \rightarrow \infty.$$

We may suppose $\tau_k \rightarrow \tau_0$ where $\tau_0 \in [1, \delta]$. We may also assume $\rho(A_k, \tau_k) \rightarrow \hat{\rho}_0$ where $\hat{\rho}_0 \in [0, \rho_0]$.

Since $\tau - 1 = \int_\rho^\varphi f'(s)ds/f$ and $f'/f' \circ \varphi - 1 = \int_\rho^\varphi \kappa f ds/f' \circ \varphi$, we have

$$p(A, \tau) = \frac{1}{\frac{f \int_\rho^\varphi \kappa f ds}{f' \circ \varphi \int_\rho^\varphi f' ds} \tau^n + \tau^{n-1} + \dots + 1}. \quad (3.13)$$

Thus, we obtain

$$a = \frac{f'^2(\hat{\rho}_0)}{\kappa(\hat{\rho}_0) f^2(\hat{\rho}_0) \tau_0^n + (\tau_0^{n-1} + \dots + 1) f'^2(\hat{\rho}_0)}.$$

If $\tau_0 > 1$, by Lemma 3.1, $\hat{\rho}_0 = 0$ and $a \geq 1/(n\delta)$. If $\tau_0 = 1$, then $a \geq a_0$. Similar arguments yield the right hand side of (3.12). \square

Let the function $\Phi(v_1, \dots, v_n)$ be given by (2.44). It is said that *the Baker-Ericksen inequalities* hold if

$$\frac{v_i \Phi_i - v_j \Phi_j}{v_i - v_j} \geq 0 \quad \text{for } i \neq j, \quad v_i \neq v_j.$$

We further assume that W is bounded below. By Theorem 2.3, the radical deformation φ , given by (2.61), with $\varphi(0) = A > 0$, is an equilibrium solution if and only if

$$\frac{v^{n-1}}{(v^n - 1)^2} \hat{\Phi}'(v) \in L^1(\delta, \infty) \quad \text{for } \delta > 1. \quad (3.14)$$

Let p be given by (2.64). Then the radical component of the Cauchy stress tensor is given by

$$T = \tau^{1-n} \Phi_1 - p \quad \text{for } \rho > 0.$$

Let $\rho_0 \in (0, 1]$ be given in Lemma 3.1. It follows from (2.64) and Lemma 3.1 (ii) that $T(0) = \lim_{\rho \rightarrow 0+} T$ exists if and only if the integral

$$\int_{\tau(\rho_0)}^\infty \frac{f' \circ \varphi}{f' \tau^n - f' \circ \varphi} \hat{\Phi}'(\tau) d\tau \quad (3.15)$$

converges.

The total stored energy of the deformation is given by

$$E(A) = \omega_n \int_0^1 f^{n-1}(\rho) \Phi(\rho) d\rho, \quad (3.16)$$

where ω_n is the area of the unit sphere S_o in M_o and $\Phi(\rho) = \Phi(\varphi'(\rho), \tau(\rho), \dots, \tau(\rho))$. We define $E(0) = \omega_n \sigma(1) \Phi(1)$.

Proposition 3.1 *Let (M, g, o) be a model with $\mu_+(1) \leq 1$ and let (3.14) hold. Let (2.61) be an equilibrium solution with $A > 0$ and $\mu_+(\varphi(1)) \leq 1$. Then*

- (i) *Then $T(0)$ exists and is finite if and only if $E(A) < \infty$.*
- (ii) *Let Φ satisfy the Baker-Ericksen inequalities. Then T is an increasing function in $\rho > 0$.*

Proof Using equation (2.65), we have

$$T'(\rho) = \frac{f' \circ \varphi}{f} \tau^{1-n} \hat{\Phi}'(\tau) = (n-1) \frac{f' \circ \varphi}{f \tau^{2n-1}} (\tau^n - 1) \frac{\tau \Phi_2 - \tau^{1-n} \Phi_1}{\tau - \tau^{1-n}} \geq 0,$$

that is, (ii) is true. Using the formula above, we obtain

$$[f^n \Phi]' = n f' f^{n-1} \Phi + f^n \hat{\Phi}'(\tau) \tau' = n f' f^{n-1} \Phi + (f^n - \frac{f' f^n \circ \varphi}{f' \circ \varphi}) T'.$$

Thus,

$$\begin{aligned} n \int_{\rho}^1 f' f^{n-1} \Phi d\rho &= f^n(1) \Phi(1) - f^n(\rho) \Phi(\rho) + \int_{\rho}^1 (\frac{f' f^n \circ \varphi}{f' \circ \varphi} - f^n) T' d\rho \\ &= f^n(1) \Phi(1) - f^n(\rho) \Phi(\rho) + (\frac{f' f^n \circ \varphi}{f' \circ \varphi} - f^n) T \Big|_{\rho=1} \\ &\quad - (\frac{f' f^n \circ \varphi}{f' \circ \varphi} - f^n) T + \int_{\rho}^1 (\frac{f' f^n \circ \varphi}{f' \circ \varphi} - f^n)_{\rho} T d\rho, \end{aligned} \quad (3.17)$$

for $0 \leq \rho \leq 1$. Using (2.15), we have

$$\lim_{\rho \rightarrow 0+} \frac{1}{f(\rho)} (\frac{f' f^n \circ \varphi}{f' \circ \varphi} - f^n)_{\rho} = -\kappa(o) f^n(A). \quad (3.18)$$

In addition, it follows from (2.64) that

$$\begin{aligned} |fT| &\leq \int_{\rho}^1 f' \circ \varphi \tau^{1-n} |\hat{\Phi}'(\tau)| d\rho = \int_{\rho_0}^1 f' \circ \varphi \tau^{1-n} |\hat{\Phi}'(\tau)| d\rho \\ &\quad + \int_{\tau(\rho_0)}^{\infty} \frac{f \circ \varphi f' \circ \varphi}{\tau (f' \tau^n - f' \circ \varphi)} |\hat{\Phi}'(\tau)| d\tau. \end{aligned}$$

Thus, by (3.14), $f|T|$ is bounded above. Therefore the last integral in the right hand side of (3.17) converges by (3.18). Moreover, $-f^n \Phi$ is also bounded above since Φ is bounded below. Thus, the left hand side of (3.17) exists and is finite if and only if $T(0)$ exists and is finite. Then (i) follows by (2.21). \square

We consider the case when there is a force acting on the unit geodesic sphere \mathbf{S} . A function on M is said to be a *force density*. We say that $\mathbf{u} \in W^{1,1}(\mathbf{B}, M)$ is an *equilibrium solution* to the boundary value problem with corresponding pressure p if $\det d\mathbf{u} = 1$ a.e. in \mathbf{B} ,

$$\partial W(d\mathbf{u}) / \partial F_{ij} - p(x) (\text{adj } d\mathbf{u})_{ij} \in L^1(\mathbf{B}) \quad \text{for } 1 \leq i, j \leq n,$$

and

$$\int_{\mathbf{B}} \langle D_F W - p D_F \det d\mathbf{u}, DH \rangle dg - \int_{\mathbf{S}} \langle Dq, H \rangle d\mathbf{S} = 0 \quad \text{for } H \in C^\infty(\bar{\mathbf{B}}, \Gamma(\zeta)),$$

where q is a given force density which is a differentiable function on M . They are the Euler-Lagrange equations for the functional

$$I_1(\mathbf{u}) = \int_{\mathbf{B}} \{W(d\mathbf{u}) - p(\det d\mathbf{u} - 1)\} dg - \int_{\mathbf{S}} q(\mathbf{u}) d\mathbf{S}.$$

A force density q is said to be *radical with respect to o* if there is $\hat{q} \in C^1(\mathbb{R})$ such that

$$q(x) = \hat{q}(\rho(x)) \quad \text{for } x \in M.$$

In particular, we take

$$q(x) = P\rho(x) \quad \text{for } x \in M,$$

where $P \in \mathbb{R}$ is a constant.

It is easy to check that the identity deformation $\varphi(\rho) = \rho$ is an equilibrium solution to the boundary value problem with the corresponding pressure

$$p = \Phi_1(1) - P.$$

Let all the assumptions in Theorem 2.3 hold. Let φ be given by (2.61) with $A > 0$. By similar arguments as in the proof of Theorem 2.3 that φ is an equilibrium solution to the above boundary value problem if and only if it is an equilibrium solution with the corresponding pressure

$$p = \tau^{1-n}\Phi_1 - T,$$

where

$$T = \frac{P}{\tau^{n-1}(1)} - \int_{\rho}^1 \frac{f' \circ \varphi}{f \circ \varphi} \tau^{2-n} \hat{\Phi}'(\tau) d\rho \quad (3.19)$$

for $\rho > 0$ and $\lambda > 0$. As a natural boundary condition given in [2] to get a unique solution for the corresponding $\lambda \in \mathbb{R}$, we assume that

$$T(0) = 0. \quad (3.20)$$

The total energy of the deformation (2.61) with $\varphi(0) = A \geq 0$ is given by

$$I(A) = \int_{\mathbf{B}} W(d\mathbf{u}) dg - P \int_{\mathbf{S}} \rho(\mathbf{u}) d\mathbf{S} = E(A) - \omega_n f^{n-1}(1) P \varphi(1), \quad (3.21)$$

where $E(A)$ is given by (3.16).

By (3.19) the possible values of A such that (3.20) is satisfied are the roots of the equation

$$P = \chi(A) \quad (3.22)$$

where

$$\chi(A) = \tau^{n-1}(1) \int_0^1 \frac{f' \circ \varphi}{f \circ \varphi} \tau^{2-n} \hat{\Phi}'(\tau) d\rho. \quad (3.23)$$

Thus, the bifurcation from the trivial solution is governed by the behavior of $\chi(A)$ as $A \rightarrow 0+$.

Let

$$\hat{I}(A) = \int_0^1 f^{n-1}(\rho) \Phi(\rho) d\rho.$$

Let $\rho_0 \in (0, 1]$ be given by Lemma 3.1. Since

$$\begin{aligned} \hat{I}(A) &= \int_{\rho_0}^1 f^{n-1}(\rho) \Phi(\rho) d\rho + \sigma(\rho_0) \hat{\Phi}(\tau(\rho_0)) - \int_0^{\rho_0} \sigma(\rho) \hat{\Phi}'(\tau) \tau' d\rho \\ &= \int_{\rho_0}^1 f^{n-1}(\rho) \Phi(\rho) d\rho + \sigma(\rho_0) \hat{\Phi}(\tau(\rho_0)) + \int_{\tau(\rho_0)}^\infty \sigma(\rho(A, \tau)) \hat{\Phi}'(\tau) d\tau, \end{aligned} \quad (3.24)$$

using the formula in (3.4) we have

$$\begin{aligned} \hat{I}'(A) &= \int_{\rho_0}^1 f^{n-1}(\rho) \hat{\Phi}'(\tau(\rho)) \tau_A(\rho) d\rho + \int_{\tau(\rho_0)}^\infty f^{n-1} \rho_A \hat{\Phi}'(\tau) d\tau \\ &= f^{n-1}(A) \left[\int_{\rho_0}^1 \frac{f' \circ \varphi}{f \circ \varphi} \tau^{2-n} \hat{\Phi}'(\tau) d\rho + \int_{\tau(\rho_0)}^\infty \frac{f' \circ \varphi}{f' \tau^n - f' \circ \varphi} \hat{\Phi}'(\tau) d\tau \right] \\ &= f^{n-1}(A) \int_0^1 \frac{f' \circ \varphi}{f \circ \varphi} \tau^{2-n} \hat{\Phi}'(\tau) d\rho \end{aligned}$$

which yields

$$I'(A) = \omega_n f^{n-1}(A) \tau^{1-n}(1) [\chi(A) - P] \quad \text{for } A \geq 0. \quad (3.25)$$

We suppose

$$\frac{\hat{\Phi}'(\tau)}{\tau^n - 1} \in L^1(\delta, \infty) \quad \text{for } \delta > 1. \quad (3.26)$$

Let $\hat{\Phi}(v)$ be twice differentiable at $v = 1$. Thus, (3.26) and $\hat{\Phi}'(1) = 0$ imply

$$\frac{\hat{\Phi}'(\tau)}{\tau^n - 1} \in L^1(1, \infty).$$

Using (3.5), we have

$$\left| \int_{\tau(\rho_0)}^\infty \sigma(\rho(A, \tau)) \hat{\Phi}'(\tau) d\tau \right| \leq c A^n \int_1^\infty \frac{1}{\tau^n - 1} |\hat{\Phi}'(\tau)| d\tau.$$

By (3.24) and (3.21), we obtain

$$\lim_{A \rightarrow 0+} I(A) = \omega_n [\sigma(1) \Phi(1) - f^{n-1}(1) P] = I(0). \quad (3.27)$$

Lemma 3.3 *Let χ be given by (3.23) and let $\hat{\Phi}(v)$ be twice differentiable at $v = 1$. Then*

$$\lim_{A \rightarrow 0+} \chi(A) = \int_1^\infty \frac{1}{\tau^n - 1} \hat{\Phi}'(\tau) d\tau. \quad (3.28)$$

Proof Let $\rho_0 \in (0, 1]$ be given in Lemma 3.1. We have

$$\int_0^1 \frac{f' \circ \varphi}{f \circ \varphi} \tau^{2-n} \hat{\Phi}'(\tau) d\rho = \int_{\rho_0}^1 \frac{f' \circ \varphi}{f \circ \varphi} \tau^{2-n} \hat{\Phi}'(\tau) d\rho + \int_{\tau(\rho_0)}^\infty \frac{f' \circ \varphi}{f' \tau^n - f' \circ \varphi} \hat{\Phi}'(\tau) d\tau, \quad (3.29)$$

for $0 < A \leq \sigma^{-1}(\rho_1 - \sigma(\rho_0))$. Since $\Phi'(1) = 0$, the first integral in the right hand side of (3.29) goes to zero as $A \rightarrow 0 +$.

Since

$$\hat{\Phi}'(\tau)/(\tau - 1) \rightarrow \Phi''(1) \quad \text{as } \tau \rightarrow 0+,$$

we fixed $\delta_0 \geq \delta > 1$ such that $|\hat{\Phi}'(\tau)|/(\tau - 1) \leq |\hat{\Phi}''(1)| + 1$ for all $1 \leq \tau \leq \delta$. By Lemma 3.2,

$$\int_{\tau(\rho_0)}^\delta \frac{f' \circ \varphi}{f' \tau^n - f' \circ \varphi} |\hat{\Phi}'(\tau)| d\tau \leq c[|\hat{\Phi}''(1)| + 1],$$

for $A > 0$ small. Thus, (3.28) follows from the dominated convergence theorem. \square

Let

$$P_{cr} = \int_1^\infty \frac{1}{\tau^n - 1} \hat{\Phi}'(\tau) d\tau.$$

The physical meaning of P_{cr} is given in [3], [2]: (3.25), (3.27) and Lemma 3.3 show that the trivial solution $A = 0$ is a local minimum (resp. local maximum) if $P < P_{cr}$ (resp. $P > P_{cr}$). The following proposition illustrates the close relations among the radial curvatures, the constitutive function $\hat{\Phi}(v)$, and the behavior of $\chi(A)$ as $A \rightarrow 0 +$.

Proposition 3.2 *Let $\hat{\Phi}(v)$ be twice differentiable at $v = 1$. Then $\chi'(0) = 0$. Furthermore, the following holds.*

(i) *Let $n = 2$. Let Φ satisfy the Baker-Ericksen inequalities with $\hat{\Phi}''(1) > 0$. Suppose there is $\varepsilon \in (0, 1]$ such that the radial curvature κ is a constant κ_0 for all $\rho \in (0, \varepsilon]$. If $\kappa_0 = 0$, then*

$$P_{cr} - \frac{1}{2} \left[1 + \frac{f^2(1)}{f'(1)} \int_\varepsilon^1 \frac{\kappa}{f} d\rho \right] \hat{\Phi}''(1) > 0 \quad (\text{resp. } < 0) \quad (3.30)$$

implies $\chi'(A) > 0$ (resp. < 0) for $A > 0$ small. If $\kappa_0 \neq 0$, then

$$\kappa_0 > 0 \quad (\text{resp. } < 0) \quad (3.31)$$

implies $\chi'(A) < 0$ (resp. > 0) for $A > 0$ small.

(ii) *Let $n \geq 3$. Let $\kappa(o) \neq 0$. Then*

$$\kappa(o) \int_1^\infty \frac{(\tau^2 - 1)\tau^n}{(\tau^n - 1)^{2(1+1/n)}} \hat{\Phi}'(\tau) d\tau < 0 \quad (\text{resp. } > 0) \quad (3.32)$$

implies $\chi'(A) > 0$ (resp. < 0) for $A > 0$ small. In addition, if there is some $\varepsilon \in (0, 1]$ such that $\kappa = 0$ for $\rho \in (0, \varepsilon)$, then

$$P_{cr} - \frac{1}{n(n-1)} \left[1 + \frac{f^n(1)}{f'(1)} \int_\varepsilon^1 \frac{\kappa}{f^{n-1}} d\rho \right] \hat{\Phi}''(1) > 0 \quad (\text{resp. } < 0) \quad (3.33)$$

implies $\chi'(A) > 0$ (resp. < 0) for $A > 0$ small.

Proof Let $\rho_0 \in (0, 1]$ be given in Lemma 3.1. Using (3.29) and (3.4), we have

$$\begin{aligned}\chi'(A) &= (n-1) \frac{\tau_A(1)}{\tau(1)} \chi(A) + \tau^{n-1}(1) \left[\int_{\rho_0}^1 \left(\frac{f' \circ \varphi}{f \circ \varphi} \tau^{2-n} \right)_A \hat{\Phi}'(\tau) d\rho \right. \\ &\quad \left. + \int_{\rho_0}^1 I_1 \hat{\Phi}''(\tau) \tau_A d\rho \right] + \tau^{n-1}(1) \left[-p(A, \tau) \frac{\hat{\Phi}'(\tau)}{\tau-1} \right]_{\tau=\tau(\rho_0)} \tau_A(\rho_0) \\ &\quad + f^{n-1}(A) \int_{\tau(\rho_0)}^{\infty} I_2 \hat{\Phi}'(\tau) d\tau, \end{aligned} \quad (3.34)$$

where

$$I_1 = \frac{f' \circ \varphi}{f \circ \varphi} \tau^{2-n}, \quad I_2 = \frac{(\kappa f'^2 \circ \varphi - \kappa \circ \varphi f'^2 \tau^2) \tau^n}{f^{n-2} (f' \tau^n - f' \circ \varphi)^3},$$

and $p(A, \tau)$ is given by (3.11). Clearly, all the terms in the right hand side of (3.34) go to zero if the last term converges to zero as $A \rightarrow 0+$.

Let $\delta > 1$ be given. It follows from (3.5) that

$$A \leq \frac{\rho_0}{c_0} (\tau^n - 1)^{1/n} \leq c(\tau - 1)^{1/n} \quad \text{for } (A, \tau) \in (0, b) \times [\tau(\rho_0), \delta]. \quad (3.35)$$

Using (3.5), (3.7), (3.12), and (3.35), we have

$$\begin{aligned}|I_2| &= \left| \frac{-f'^2 \circ \varphi \int_{\rho}^{\varphi} \kappa'(s) ds + \kappa \circ \varphi [f'^2 \circ \varphi (1 - \tau^2) + 2\tau^2 \int_{\rho}^{\varphi} \kappa f f' ds]}{f^{n-2} f'^3 \circ \varphi (\tau - 1)^3} \right| p^3 \tau^n \\ &\leq c \frac{A(\tau - 1)^{1-1/n} + \tau - 1}{A^{n-2} (\tau - 1)^{2(1+1/n)}} \leq \frac{c}{A^{n-2} (\tau - 1)^{1+2/n}} \quad \text{for } (A, \tau) \in (0, b) \times [\tau(\rho_0), \delta]. \end{aligned}$$

Thus, by (3.35) again,

$$f^{n-1}(A) |I_2 \hat{\Phi}'(\tau)| \leq c \frac{A^{1-\alpha}}{(\tau - 1)^{(2-\alpha)/n}} \left| \frac{\hat{\Phi}'(\tau)}{\tau - 1} \right| \quad \text{for } (A, \tau) \in (0, b) \times [\tau(\rho_0), \delta] \quad (3.36)$$

and for $0 \in [0, 1]$. It follows from (3.34) and (3.36) that $\chi'(0) = \lim_{A \rightarrow 0+} \chi'(A) = 0$.

(i) We may assume that $0 < \rho_0 < \varepsilon$ is small enough such that $\varphi \leq \varepsilon$ when $\rho \in (0, \rho_0]$. Let $\kappa_0 = 0$. Then $f'(\rho_0) = 1$, $f(\rho_0) = \rho_0$, and $I_2 = 0$. In this case we have, by (3.34) and (3.13),

$$\begin{aligned}\lim_{A \rightarrow 0+} \frac{\chi'(A)}{f(A)} &= \frac{f'(1)}{f^2(1)} P_{cr} + \lim_{A \rightarrow 0+} \int_{\rho_0}^1 \frac{f' \circ \varphi}{f f \circ \varphi} I_1 \hat{\Phi}''(\tau) d\rho - \lim_{A \rightarrow 0+} \frac{p(A, \tau) f' \circ \varphi}{f f \circ \varphi} \frac{\hat{\Phi}'(\tau)}{\tau - 1} \Big|_{\tau=\tau(\rho_0)} \\ &= \frac{f'(1)}{f^2(1)} P_{cr} + \left(\int_{\rho_0}^1 \frac{f'^2}{f^3} d\rho - \frac{1}{2\rho_0^2} \right) \hat{\Phi}''(1) \\ &= \frac{f'(1)}{f^2(1)} P_{cr} - \frac{1}{2} \left[\frac{f'(1)}{f^2(1)} + \int_{\varepsilon}^1 \frac{\kappa}{f} d\rho \right] \hat{\Phi}''(1). \end{aligned}$$

Thus, the case $\kappa_0 = 0$ follows.

Let $\kappa_0 \neq 0$. Let $q = f'/f \circ \varphi$. By (3.2), $(f'/f)' = -(\kappa f^2 + f'^2)/f^2 < 0$ for $\rho \in (0, \rho_0]$. Thus,

$$q^2 \tau^2 > 1 \quad \text{for } (A, \tau) \in (0, b] \times [\tau(\rho_0), \infty).$$

Let $\delta > 1$ be fixed such that

$$\frac{1}{\tau-1}\hat{\Phi}'(\tau) \geq \frac{1}{2}\hat{\Phi}''(1) \quad \text{for } \tau \in [1, \delta].$$

We have

$$\left| \frac{(q^2-1)\tau^2}{\tau-1} + \tau + 1 \right| \geq 2 - \left| \frac{2\kappa_0 f \int_{\rho}^{\varphi} f f' ds}{f'^2 \circ \varphi \int_{\rho}^{\varphi} f' ds} \right| \tau^2 \geq 2 - c|\kappa_0|f^2(\rho_0),$$

for $(A, \tau) \in (0, b] \times [\tau(\rho_0), \delta]$. We assume that $\rho_0 \in (0, \varepsilon]$ is also such that $2 - c|\kappa_0|f^2(\rho_0) > 0$. Thus, by (3.12), we obtain

$$-\frac{1}{\kappa_0}I_2\hat{\Phi}' = \frac{(q^2\tau^2-1)\tau^2p(A, \tau)}{f' \circ \varphi(\tau-1)^3}\hat{\Phi}'(\tau) \geq \frac{c_0}{\tau-1}\hat{\Phi}''(1) \quad \text{for } (A, \tau) \in (0, b] \times [\tau(\rho_0), \delta],$$

which yields, by (3.34),

$$\lim_{A \rightarrow 0+} \frac{\chi'(A)}{f(A)} = \begin{cases} -\infty & \text{if } \kappa_0 > 0; \\ +\infty & \text{if } \kappa_0 < 0. \end{cases}$$

Thus, the case $\kappa_0 \neq 0$ follows.

(ii) Since $n \geq 3$, (3.36) with $\alpha = 0$ and (3.26) imply the integral

$$\int_{\tau(\rho_0)}^{\cdot} f^{n-2}(A)I_2\hat{\Phi}'(\tau)d\tau$$

converges as $A \rightarrow 0+$. Using (3.34), (3.5), (3.6), and (3.8), we obtain

$$\lim_{A \rightarrow 0+} \frac{\chi'(A)}{f(A)} = \kappa(o) \int_1^{\infty} \frac{(1-\tau^2)\tau^n}{(\tau^n-1)^{2(1+1/n)}} d\tau,$$

which gives (3.32). Finally, a similar computation as in (i) for the case $\kappa_0 = 0$ yields (3.33). \square

Remark 3.1 *If Φ satisfies the Baker-Ericksen inequalities, then (3.32) is equivalent to*

$$\kappa(o) < 0 \quad (\text{resp. } > 0).$$

Let $P > 0$. If A_0 is a root of (3.22), that is,

$$P = \chi(A_0),$$

then from (3.25)

$$I''(A_0) = \omega_n f^{n-1}(A_0) \tau^{1-n}(1) \chi'(A_0).$$

Let Φ satisfy the Baker-Ericksen inequalities and let $n \geq 3$. From (3.32), A_0 is a local minimum (resp. local maximum) of I if $\kappa(o) < 0$ (resp. $\kappa(o) > 0$) (for A_0 small).

4 Cavitation in the Compressible Case

By Theorem 2.1, an equilibrium solution φ satisfies the equation

$$[f^{n-1}(\rho)\Phi_1(\rho)]_\rho = (n-1)f^{n-2}(\rho)f' \circ \varphi(\rho)\Phi_2(\rho) \quad \text{for } x \in \Omega, \quad \rho(x) > 0, \quad (4.1)$$

where

$$\Phi_1(\rho) = \Phi_1(\varphi', \tau, \dots, \tau), \quad \Phi_2(\rho) = \Phi_2(\varphi', \tau, \dots, \tau), \quad \tau(\rho) = \frac{f \circ \varphi(\rho)}{f(\rho)}.$$

Let φ be a solution to problem (4.1) with $\varphi' > 0$ and $\varphi > 0$ on $(0, 1]$. We define

$$T(\rho) = \tau^{n-1}(\rho)\Phi_1(\rho) \quad \text{for } \rho \in (0, 1], \quad (4.2)$$

which is the radial component of the Cauchy stress ([2]). By (4.1),

$$T'(\rho) = (n-1)\frac{f' \circ \varphi}{f \circ \varphi}\tau^{1-n}(\tau\Phi_2 - \varphi'\Phi_1) \quad \text{for } \rho \in (0, 1]. \quad (4.3)$$

It follows from (4.3) that

Proposition 4.1 *Let φ be a solution of (4.1) with $\varphi'(\rho) > 0$ for all $\rho \in (0, 1]$. If the Baker-Ericksen inequalities hold, then*

$$T'(\rho)[\varphi'(\rho) - \tau(\rho)] \leq 0 \quad \text{for } \rho \in (0, 1]. \quad (4.4)$$

Let

$$\tilde{T}(\rho) = \Phi(\rho) - \varphi'(\rho)\Phi_1(\rho), \quad (4.5)$$

where $\Phi(\rho) = \Phi(\varphi', \tau, \dots, \tau)$. \tilde{T} is said to be the radial component of the inverse Cauchy stress ([2]). We obtain by (4.1) and (4.3)

$$\tilde{T}'(\rho) = -\frac{f'}{f' \circ \varphi}\tau^n T' \quad \text{for } \rho \in (0, 1]. \quad (4.6)$$

It follows (4.6), (4.3) and (4.1) that

$$\begin{aligned} \left\{ f^n[\Phi - (\varphi' - \tau)\Phi_1] \right\}' &= [f^n\tilde{T} + f \circ \varphi f^{n-1}\Phi_1]' = n f' f^{n-1}\tilde{T} + f^n\tilde{T}' \\ &\quad + f' \circ \varphi \varphi' f^{n-1}\Phi_1 + f \circ \varphi (f^{n-1}\Phi_1)' \\ &= n f' f^{n-1}\Phi + (f' \circ \varphi - f') f^{n-1}[\varphi'\Phi_1 + (n-1)\tau\Phi_2]. \end{aligned} \quad (4.7)$$

If $\kappa = 0$ for $\rho \in (0, 1]$, then $f' = 1$, $f = \rho$, and (4.7) becomes

$$\left\{ \rho^n[\Phi - (\varphi' - \tau)\Phi_1] \right\}' = n\rho^{n-1}\Phi, \quad \tau = \frac{\varphi}{\rho},$$

which is the radial version of the conservation law ([2]).

4.1 Constitutive Assumptions

Throughout this paper unless otherwise stated we assume the class of constitutive functions $W(F) = \Phi(v_1, \dots, v_n)$ have the form ($n \geq 2$)

$$\Phi(v_1, \dots, v_n) = \sum_{i=1}^n \phi(v_i) + h(v_1 \cdots v_n), \quad (4.8)$$

where functions ϕ and h satisfy the following assumptions:

(A 1) $h : (0, \infty) \rightarrow \mathbb{R}$ is C^2 and strictly convex;

(A 2) $\lim_{v \rightarrow 0+} h(v) = \lim_{v \rightarrow \infty} \frac{h(v)}{v} = +\infty$;

(A 3) h satisfies

$$\lim_{v \rightarrow \infty} \frac{vh'(v)}{h(v)} > 1,$$

and let

$$\theta(s) = \lim_{v \rightarrow \infty} \frac{h(sv)}{h(v)} \quad \text{for } s \in (0, \infty),$$

and we assume that $\theta : (0, \infty) \rightarrow (0, \infty)$ is continuous;

(A 4) $\phi : (0, \infty) \rightarrow (0, \infty)$ is C^2 and convex;

(A 5) $v\phi'(v)$ is increasing on $(0, \infty)$;

(A 6) Let $t_0 \geq 0$ be such that $\phi'(t_0) = 0$. Let

$$q_1(s) = \sup_{v > t_0} \frac{\phi'(v)}{\phi'(sv)} \quad \text{for } s > 1; \quad q_0(s) = \inf_{v > t_0/s} \frac{\phi'(v)}{\phi'(sv)} \quad \text{for } s \in (0, 1].$$

We assume that $q_1 \in C^1[1, \infty)$ and $q_0 \in C^1(0, 1]$ satisfy

$$\lim_{s \rightarrow \infty} q_1(s) = 0, \quad \lim_{s \rightarrow 0+} q_0(s) = \infty, \quad (4.9)$$

$$q_1'(s) < 0 \quad \text{for } s \in [1, \infty) \quad \text{and} \quad q_0'(s) < 0 \quad \text{for } s \in (0, 1], \quad (4.10)$$

respectively.

(A 7) there are $\delta_0 > 0$ and $\delta_1 > 0$ such that if $|s - 1| < \delta_0$ then

$$|\phi'(sv)| \leq \delta_1 \frac{\phi(v)}{v} \quad \text{for all } v > 0;$$

(A 8) $\phi(v) \leq \delta_2(1 + v^\alpha + v^{-\beta})$ for all $v > 0$, where $\delta_2 > 0$, $0 < \alpha < n$, and $0 \leq \beta < 1 + 1/(n - 1)$;

(A 9) $\lim_{v \rightarrow \infty} \phi(v) = +\infty$.

Remark 4.1 (A1), (A2), (A4), (A5), (A7), (A8), and (A9) are given in [2]. In addition, it is easy to check that q_1 and q_0 are decreasing on $[1, \infty)$ and $(0, 1]$, respectively, and $q_1(1) = q_0(1) = 1$.

Example 4.1 *Let*

$$\phi(v) = \mu(v^\alpha - n) + \frac{\nu}{v^\beta},$$

where $\mu > 0$, $\nu \geq 0$, $1 < \alpha < n$, and $0 \leq \beta < 1 + 1/(n-1)$. Let

$$h(v) = H(v) - n,$$

where $H : (0, \infty) \rightarrow \mathbb{R}^+$ is a C^3 function and satisfies $\lim_{\delta \rightarrow 0+} H(\delta) = \infty$, $H''(\delta) > 0$ for all $\delta > 0$, and

$$H(\delta) = k(\delta - 1 - k^{-1})^2 \quad \text{for } \delta \geq 1/2,$$

where $k > 1$.

Clearly, (A4), (A5), (A7), (A8), and (A9) are true for ϕ . We check (A6). Since

$$\frac{\phi'(v)}{\phi'(sv)} = \frac{h_1(v)}{h_2(v)},$$

where

$$h_1(v) = \mu\alpha - \frac{\beta\nu}{v^{\alpha+\beta}}, \quad h_2(v) = \mu\alpha s^{\alpha-1} - \frac{\beta\nu}{s^{\beta+1}v^{\alpha+\beta}},$$

$$h'_1(v)h_2(v) - h'_2(v)h_1(v) = \frac{\beta(\alpha+\beta)\nu\mu\alpha}{v^{\alpha+\beta+1}}(s^{\alpha-1} - \frac{1}{s^{\alpha+\beta}}),$$

we have

$$q_1(s) = \lim_{v \rightarrow \infty} \frac{h_1(v)}{h_2(sv)} = \frac{1}{s^{\alpha-1}} \quad \text{for } s \in [1, \infty); \quad q_0(s) = \frac{1}{s^{\alpha-1}} \quad \text{for } s \in (0, 1].$$

Thus, (A6) holds.

(A1) and (A2) automatically hold for h . (A3) is also true since

$$\lim_{v \rightarrow \infty} \frac{vh'(v)}{h(v)} = 2, \quad \theta(s) = s^2 \quad \text{for } s \in (0, \infty).$$

4.2 Equilibrium Solutions

Let $\lambda > 0$ be given. It follows from (4.1) and (4.8) that equilibrium solutions are given by problem

$$\begin{cases} \varphi(1) = \lambda, \\ f[\phi''(\varphi') + h''(\varphi'\tau^{n-1})\tau^{2(n-1)}]\varphi'' = (n-1)[f' \circ \varphi\phi'(\tau) - f'\phi'(\varphi')] \\ \quad -(n-1)(f' \circ \varphi\varphi' - f'\tau)h''(\varphi'\tau^{n-1})\varphi'\tau^{2n-3} \quad \text{for } \rho \in (0, 1). \end{cases} \quad (4.11)$$

Regular Equilibrium Solutions An equilibrium solution $\varphi \in C^1(0, 1]$ to problem (4.11) is said to be *regular* if $\varphi(0) = \lim_{\rho \rightarrow 0+} \varphi(\rho) = 0$. In the case of the Euclidean space there is a unique regular equilibrium solution $\varphi = \lambda\rho$ which plays an important role in the analysis ([2, 10]). We derive some properties of regular equilibrium solutions.

We assume that $\lambda > 0$ is given such that the radial curvature satisfies

$$\mu_+(\max\{\lambda, 1\}) = \int_0^{\max\{\lambda, 1\}} s\kappa_+(s)ds \leq 1. \quad (4.12)$$

By Proposition 2.4

$$f'(\rho) > 0 \quad \text{for } \rho \in [0, \max\{\lambda, 1\}].$$

Let

$$b_0(\rho) = \min_{0 \leq s \leq \rho} f'(s), \quad b_1(\rho) = \sup_{0 \leq s \leq \rho} f'(s) \quad \text{for } \rho \in [0, \max\{\lambda, 1\}].$$

Set

$$\begin{aligned} \alpha_1(\rho, s) &= \max \left\{ b_1(\rho)/b_0(s), \quad q_1^{-1}(b_0(\rho)/b_1(s)) \right\}, \\ \alpha_0(\rho, s) &= \min \left\{ b_0(\rho)/b_1(s), \quad q_0^{-1}(b_1(\rho)/b_0(s)) \right\} \end{aligned}$$

for $(\rho, s) \in [0, 1] \times [0, \lambda]$, where q_0 and q_1 are given by (A 6).

Then the following lemma is immediate.

Lemma 4.1 *Let $\mu_+(\max\{\lambda, 1\}) \leq 1$ and let $(\rho, s) \in [0, 1] \times [0, \lambda]$ be given. $\alpha_1(\cdot, s)$ and $\alpha_1(\rho, \cdot)$ are increasing. $\alpha_0(\cdot, s)$ and $\alpha_0(\rho, \cdot)$ are decreasing.*

We have

Theorem 4.1 *Let $\mu_+(\max\{\lambda, 1\}) \leq 1$. Let (A1), (A4), (A5), and (A6) hold. If $\varphi \in C^1(0, 1]$ is a regular equilibrium solution to problem (4.11), then*

$$\alpha_0(\rho, \varphi(\rho))\tau(\rho) \leq \varphi'(\rho) \leq \alpha_1(\rho, \varphi(\rho))\tau(\rho) \quad \text{for } \rho \in (0, 1]. \quad (4.13)$$

Proof First, we prove that the right hand side of the inequalities (4.13) holds true. We suppose for a contradiction that there is $\rho_0 \in (0, 1]$ such that

$$\varphi'(\rho_0) > \alpha_1(\rho_0, \varphi(\rho_0))\tau(\rho_0). \quad (4.14)$$

Since $\alpha_1(\rho_0, \varphi(\rho_0)) \geq b_1(\rho_0)/b_0 \circ \varphi(\rho_0)$, by (4.14) we have

$$f\tau' = f' \circ \varphi\varphi' - f'\tau > (b_0\alpha_1 - b_1)\tau \geq 0 \quad \text{at } \rho = \rho_0. \quad (4.15)$$

Next, we claim

$$f' \circ \varphi\phi'(\tau) - f'\phi'(\varphi') \leq 0 \quad \text{at } \rho = \rho_0. \quad (4.16)$$

We assume that $\phi'(\tau(\rho_0)) \leq 0$. Since ϕ' is increasing, $\alpha_1(\rho_0, \varphi(\rho_0))f' \circ \varphi(\rho_0) \geq f'(\rho_0)$, and $\alpha_1(\rho_0, \varphi(\rho_0)) \geq 1$, it follows from (4.14) and (A 5) that

$$\begin{aligned} f' \circ \varphi\phi'(\tau) - f'\phi'(\varphi') &\leq f' \circ \varphi\phi'(\tau) - f'\phi'(\alpha_1\tau) \\ &= \left[f' \circ \varphi - \frac{f'}{\alpha_1} \right] \phi'(\tau) + \frac{f'}{\alpha_1\tau} [\tau\phi'(\tau) - \alpha_1\tau\phi'(\alpha_1\tau)] \leq 0 \quad \text{at } \rho = \rho_0. \end{aligned}$$

Let $\phi'(\tau) > 0$. Let

$$t_1 = q_1^{-1}\left(\frac{b_0(\rho_0)}{b_1 \circ \varphi(\rho_0)}\right).$$

Then $q_1(t_1) = b_0(\rho_0)/b_1 \circ \varphi(\rho_0)$, that is,

$$b_1 \circ \varphi(\rho_0)\phi'(\tau(\rho_0)) \leq b_0(\rho_0)\phi'(t_1\tau(\rho_0)),$$

which implies that (4.16) holds true since $\alpha_1(\rho_0, \varphi(\rho_0)) \geq t_1$.

From (4.16), (4.15), and (4.11) we obtain

$$\varphi''(\rho_0) < 0. \quad (4.17)$$

By (4.15) and (4.17) there is the smallest number $\rho_1 \in [0, \rho_0)$ such that

$$\tau' > 0 \quad \text{and} \quad \varphi''(\rho) < 0 \quad \text{for} \quad \rho \in (\rho_1, \rho_0). \quad (4.18)$$

We claim $\rho_1 = 0$. If $\rho_1 > 0$, then by (4.18) and (4.14)

$$\varphi'(\rho_1) > \varphi'(\rho_0) > \alpha_1(\rho_0, \varphi(\rho_0))\tau(\rho_0) \geq \alpha_1(\rho_1, \varphi(\rho_1))\tau(\rho_1),$$

since $\alpha_1(\rho_0, \varphi(\rho_0)) \geq \alpha_1(\rho_1, \varphi(\rho_1))$ by $\rho_0 \geq \rho_1$ and $\varphi(\rho_0) \geq \varphi(\rho_1)$. Using the same arguments as for (4.15) and (4.16), we obtain

$$\tau'(\rho_1) > 0, \quad \varphi''(\rho_1) < 0,$$

reaching a contradiction.

It follows from (4.18) with $\rho_1 = 0$ that $\varphi'(\rho) > \varphi'(\rho_0)$ for $\rho \in (0, \rho_0)$ and

$$\varphi(\rho) < \varphi(\rho_0) - \varphi'(\rho_0)\rho_0 + \varphi'(\rho_0)\rho \quad \text{for} \quad \rho \in (0, \rho_0). \quad (4.19)$$

Moreover, (4.14) implies that

$$\frac{\rho_0\varphi'(\rho_0)}{\varphi(\rho_0)} > \alpha_1(\rho_0, \varphi(\rho_0))\frac{\rho_0}{f(\rho_0)}\frac{f \circ \varphi(\rho_0)}{\varphi(\rho_0)} \geq \alpha_1(\rho_0, \varphi(\rho_0))\frac{b_0 \circ \varphi(\rho_0)}{b_1(\rho_0)} \geq 1, \quad (4.20)$$

since $\rho b_0(\rho) \leq f(\rho) \leq b_1(\rho)\rho$ for all $\rho \geq 0$.

From (4.19) and (4.20) we obtain $\varphi(0) < 0$ which is a contradiction again the assumption (4.14).

Similar arguments prove the left hand side of the inequalities (4.13). \square

Actually, from the proof of Theorem 4.1 we have shown that

Corollary 4.1 *Let $\mu_+(\max\{\lambda, 1\}) \leq 1$. Let (A1), (A4), (A5), and (A6) hold. Let $\varphi \in C^1(0, 1]$ be an equilibrium solution to problem (4.11). Then*

$$\varphi'(\rho) \leq \alpha_1(\rho, \varphi(\rho))\tau(\rho) \quad \text{for} \quad \rho \in (0, 1].$$

Furthermore, if there is $\rho_0 \in (0, 1]$ such that

$$\varphi'(\rho_0) < \alpha_0(\rho_0, \varphi(\rho_0))\tau(\rho_0),$$

then $\varphi(0) > 0$.

Let $\kappa \equiv 0$. Then $f = \rho$, $b_0 = b_1 = 1$, and

$$\alpha_0(\rho, s) = \alpha_1(\rho, s) = 1 \quad \text{for } \rho, s \in [0, \infty).$$

It follows from (4.11) and (4.13) that a regular equilibrium solution φ satisfies

$$\begin{cases} \varphi(1) = \lambda, \\ \varphi'(\rho) = \frac{\varphi(\rho)}{\rho} \quad \text{for } \rho \in (0, 1). \end{cases} \quad (4.21)$$

Since problem (4.21) has the unique solution $\varphi = \lambda\rho$, we have

Corollary 4.2 ([10]) *Let $\kappa \equiv 0$ and let (A1), (A4), (A5), and (A6) hold. Then there exists a unique regular equilibrium solution $\varphi = \lambda\rho$ to problem (4.11) with $\varphi(1) = \lambda$.*

Let $0 < \mu_0 \leq \mu_1 < \infty$ be given by (2.31). It follows from (2.31) that

$$\frac{\mu_0}{\mu_1} \leq \frac{b_0(\rho)}{b_1(s)} \leq 1 \leq \frac{b_1(\rho)}{b_0(s)} \leq \frac{\mu_1}{\mu_0} \quad \text{for } \rho, s \in [0, \infty),$$

which also imply by (A5) that

$$q_0^{-1}\left(\frac{\mu_1}{\mu_0}\right) \leq q_0^{-1}\left(\frac{b_1(\rho)}{b_0(s)}\right) \leq 1 \leq q_1^{-1}\left(\frac{b_0(\rho)}{b_1(s)}\right) \leq q_1^{-1}\left(\frac{\mu_0}{\mu_1}\right) \quad \text{for } \rho, s \in [0, \infty).$$

It follows from Theorem 4.8 and Proposition 2.5 that

Corollary 4.3 *Let $\mu_+(\infty) \leq 1$ and $\mu_-(\infty) < \infty$. Let (A1), (A4), (A5), and (A6) hold. Let $\varphi \in C^1(0, 1]$ be a regular equilibrium solution to problem (4.11) with $\varphi(1) = \lambda$. Then*

$$\eta_0\tau(\rho) \leq \varphi'(\rho) \leq \eta_1\tau(\rho) \quad \text{for } \rho \in (0, 1], \quad (4.22)$$

where

$$\eta_0 = \min \left\{ \frac{\mu_0}{\mu_1}, q_0^{-1}\left(\frac{\mu_1}{\mu_0}\right) \right\}, \quad \eta_1 = \max \left\{ \frac{\mu_1}{\mu_0}, q_1^{-1}\left(\frac{\mu_0}{\mu_1}\right) \right\}.$$

Corollary 4.4 *Let all the assumptions in Corollary 4.3 hold. If $\varphi \in C^1(0, 1]$ is a regular equilibrium solution to problem (4.11) with $\varphi(1) = \lambda$, then*

$$\lambda\rho^{c_1} \leq \varphi(\rho) \leq \lambda\rho^{c_0} \quad \text{for } \rho \in (0, 1], \quad \lambda > 0, \quad (4.23)$$

where

$$c_0 = \eta_0 \frac{\mu_0}{\mu_1}, \quad c_1 = \eta_1 \frac{\mu_1}{\mu_0}.$$

Proof Using (2.31) and (4.22), we have

$$\frac{\varphi'}{\varphi} \leq \eta_1 \frac{f \circ \varphi}{\varphi} \frac{\rho}{f} \frac{1}{\rho} \leq \frac{c_1}{\rho} \quad \text{for } \rho \in (0, 1],$$

which yields the left hand side of the inequalities (4.23). A similar argument proves the right hand side of the inequalities (4.23). \square

Remark 4.2 If $\kappa(s) = 0$ for all $s \geq 0$, then $\mu_0 = \mu_1 = \eta_0 = \eta_1 = 1$, and (4.23) means $\varphi(\rho) = \lambda\rho$.

Theorem 4.2 Let $\mu_+(\max\{\lambda, 1\}) \leq 1$. Let (A1), (A4), (A5), and (A6) hold. Let $\varphi \in C^1(0, 1]$ be a regular equilibrium solution to problem (4.11) with $\varphi(1) = \lambda$. Then

(i) there are constants $\rho_0 \in (0, 1]$, $c_0 > 0$, and $c_1 > 0$ such that

$$c_0\rho \leq \varphi(\rho) \leq c_1\rho \quad \text{for all } 0 \leq \rho \leq \rho_0; \quad (4.24)$$

(ii) the limit $\lim_{\rho \rightarrow 0+} \varphi(\rho)/\rho = \varphi'(0)$ exists;

(iii) there are constants $\lambda_0 > 0$, $c_0 > 0$, and $c_1 > 0$ such that

$$c_0\lambda \leq \varphi'(0) \leq c_1\lambda \quad \text{for all } \lambda \in (0, \lambda_0]. \quad (4.25)$$

Proof (i) First, we prove the right hand side of inequalities (4.24).

We fix $0 < \rho_0 \leq 1$ small such that

$$e^{\mu_- \circ \varphi(\rho_0)} \mu_+ \circ \varphi(\rho_0) < 1, \quad (4.26)$$

which is possible since $\varphi(0) = 0$.

Step 1 Let

$$\alpha(\rho) = \frac{b_1(\rho)}{b_0 \circ \varphi(\rho)} \quad \text{for } \rho \in (0, \rho_0].$$

We shall estimate

$$\int_{\rho}^{\rho_0} \frac{\alpha(s)}{\varphi(s)} \tau(s) ds \quad \text{for } \rho \in (0, \rho_0].$$

Since by Proposition 2.4

$$b_0(\rho) = 1 - \sup_{0 \leq s \leq \rho} \int_0^s \kappa f ds \geq 1 - \int_0^{\rho} \kappa_+ f ds \geq 1 - e^{\mu_- (\rho)} \mu_+(\rho) \quad \text{for } \rho \in (0, 1],$$

it follows from (4.26) that

$$b_0 \circ \varphi(\rho) \geq b_0 \circ \varphi(\rho_0) > 0 \quad \text{for } 0 \leq \rho \leq \rho_0. \quad (4.27)$$

Moreover, we have

$$\begin{aligned} b_1(\rho) - b_0 \circ \varphi(\rho) &= \sup_{0 \leq s \leq \varphi(\rho)} \int_0^s \kappa f ds - \inf_{0 \leq s \leq \rho} \int_0^s \kappa f ds \\ &\leq \int_0^{\varphi(\rho)} \kappa_+ f ds + \int_0^{\rho} \kappa_- f ds \leq c(\rho_0, \varphi(\rho_0)) [f \circ \varphi(\rho) + f(\rho)] \end{aligned} \quad (4.28)$$

for $0 \leq \rho \leq \rho_0$, where

$$c(t, s) = \int_0^{\max\{t, s\}} |\kappa(\zeta)| d\zeta. \quad (4.29)$$

Using (4.13), (4.27), and (4.28) we have

$$\begin{aligned} \frac{\alpha(\rho) - 1}{\varphi(\rho)} \tau(\rho) &= \frac{\alpha(\rho) - 1}{f(\rho)} \frac{f \circ \varphi(\rho)}{\varphi(\rho)} \leq \frac{c(\rho_0, \varphi(\rho_0))}{b_0 \circ \varphi(\rho_0)} (\tau + 1) b_1 \circ \varphi(\rho_0) \\ &\leq \frac{c(\rho_0, \varphi(\rho_0)) b_1 \circ \varphi(\rho_0)}{\alpha_0(\rho_0, \varphi(\rho_0)) b_0 \circ \varphi(\rho_0)} \varphi'(\rho) + \frac{c(\rho_0, \varphi(\rho_0)) b_1 \circ \varphi(\rho_0)}{b_0 \circ \varphi(\rho_0)} \quad \text{for } 0 \leq \rho \leq \rho_0, \end{aligned} \quad (4.30)$$

since $f \circ \varphi \leq (b_1 \circ \varphi) \varphi$.

In addition we have

$$\begin{aligned} \frac{f \circ \varphi(\rho)}{\varphi(\rho)} - 1 &= - \int_0^{\varphi(\rho)} \kappa f ds + \frac{1}{\varphi(\rho)} \int_0^{\varphi(\rho)} s \kappa f ds \\ &\leq 2c(\rho_0, \varphi(\rho_0)) f \circ \varphi(\rho) \quad \text{for } 0 \leq \rho \leq \rho_0, \end{aligned}$$

which implies by (4.13) that

$$\left[\frac{f \circ \varphi(\rho)}{\varphi(\rho)} - 1 \right] \frac{1}{f(\rho)} \leq \frac{2c(\rho_0, \varphi(\rho_0))}{\alpha_0(\rho_0, \varphi(\rho_0))} \varphi'(\rho) \quad \text{for } 0 < \rho \leq \rho_0. \quad (4.31)$$

Furthermore,

$$\frac{1}{f(\rho)} - \frac{1}{\rho} = \frac{1}{f(\rho)} \left[\int_0^\rho \kappa \left(1 - \frac{s}{\rho}\right) f ds \right] \leq 2c(\rho_0, \varphi(\rho_0)) \quad \text{for } 0 < \rho \leq \rho_0. \quad (4.32)$$

From (4.30), (4.31), and (4.32), we obtain

$$\begin{aligned} \frac{\alpha(\rho)}{\varphi(\rho)} \tau(\rho) &= \frac{\alpha(\rho) - 1}{\varphi(\rho)} \tau(\rho) + \left[\frac{f \circ \varphi(\rho)}{\varphi(\rho)} - 1 \right] \frac{1}{f} + \left(\frac{1}{f(\rho)} - \frac{1}{\rho} \right) + \frac{1}{\rho} \\ &\leq c(\rho_0, \varphi(\rho_0)) \left[\frac{b_1 \circ \varphi(\rho_0)}{b_0 \circ \varphi(\rho_0)} + 2 \right] \left[\frac{\varphi'(\rho)}{\alpha_0(\rho_0, \varphi(\rho_0))} + 1 \right] + \frac{1}{\rho} \end{aligned}$$

for $0 < \rho \leq \rho_0$, which yields

$$\int_\rho^{\rho_0} \frac{\alpha(s)}{\varphi(s)} \tau(s) ds \leq \eta(\rho_0, \varphi(\rho_0)) + \ln \frac{\rho_0}{\rho} \quad \text{for } 0 < \rho \leq \rho_0, \quad (4.33)$$

where

$$\eta(t, s) = c(t, s) \left[\frac{b_1(s)}{b_0(s)} + 2 \right] \left[\frac{s}{\alpha_0(t, s)} + t \right]. \quad (4.34)$$

Step 2 Let

$$\alpha * (\rho) = \frac{b_0(\rho)}{b_1 \circ \varphi(\rho)} \quad \text{for } 0 \leq \rho \leq \rho_0.$$

By (4.10) we have

$$q_1^{-1}(\alpha * (\rho)) - 1 \leq \frac{1}{c * (\rho_0, \varphi(\rho_0))} [1 - \alpha * (\rho)] \quad \text{for } 0 \leq \rho \leq \rho_0,$$

where

$$c * (t, s) = \inf_{b_0(t)/b_1(s) \leq \zeta \leq 1} |q_1'(q_1^{-1}(\zeta))|. \quad (4.35)$$

Similar arguments as in **Step 1** give the estimate

$$\int_{\rho}^{\rho_0} q_1^{-1}(\alpha * (s)) \frac{f \circ \varphi(s)}{f(s)\varphi(s)} ds \leq \eta * (\rho_0, \varphi(\rho_0)) + \ln \frac{\rho_0}{\rho} \quad \text{for } 0 < \rho \leq \rho_0, \quad (4.36)$$

where

$$\eta * (t, s) = c(t, s) \left[\frac{b_1(s)}{c * (t, s)} + 2 \right] \left[\frac{s}{\alpha_0(t, s)} + t \right],$$

where $c(t, s)$ and $c * (t, s)$ are given by (4.29) and (4.35), respectively.

Step 3 Using (4.13), (4.33), and (4.36) we obtain

$$\varphi(\rho) \geq e^{-\max\{\eta(\rho_0, \varphi(\rho_0)), \eta * (\rho_0, \varphi(\rho_0))\}} \frac{\varphi(\rho_0)}{\rho_0} \rho \quad \text{for } 0 \leq \rho \leq \rho_0. \quad (4.37)$$

Similar arguments prove the left hand side of inequalities (4.24).

(ii) From (4.37) we have

$$\underline{\lim}_{\rho \rightarrow 0+} \frac{\varphi(\rho)}{\rho} \geq e^{-\max\{\eta(\rho_0, \varphi(\rho_0)), \eta * (\rho_0, \varphi(\rho_0))\}} \frac{\varphi(\rho_0)}{\rho_0} \quad (4.38)$$

for $\rho_0 \in (0, 1]$ such that (4.26) holds. It follows from (4.38) that

$$\underline{\lim}_{\rho \rightarrow 0+} \frac{\varphi(\rho)}{\rho} \geq \overline{\lim}_{\rho \rightarrow 0+} \frac{\varphi(\rho)}{\rho}$$

since $\lim_{\rho \rightarrow 0+} \eta(\rho, \varphi(\rho)) = \lim_{\rho \rightarrow 0+} \eta * (\rho, \varphi(\rho)) = 0$, which is what we need.

(iii) Let $\lambda_0 > 0$ be small such that

$$e^{\mu - (\lambda_0)} \mu_+(\lambda_0) < 1.$$

Let $0 < \lambda \leq \lambda_0$. Then ρ_0 in (4.26) can be taken as 1. It follows from (4.37) that

$$\varphi'(0) \geq e^{-\max\{\eta(1, \lambda), \eta * (1, \lambda)\}} \lambda \quad \text{for } \lambda \in (0, \lambda_0].$$

Since $\eta(1, \lambda)$ and $\eta * (1, \lambda)$ are bounded on $[0, \lambda]$, we have shown that the left hand side of inequalities (4.25). Similar arguments show that the right hand side of (4.25) holds. \square

Remark 4.3 In (4.24) constants c_0 and c_0 may be depend on the number λ .

Proposition 4.2 Let $\mu_+(\max\{\lambda, 1\}) \leq 1$. Let (A1), (A2), (A4), (A5), (A6), (A7) and (A8) hold. Let $\varphi \in C^1(0, 1]$ be an equilibrium solution to problem (4.11) with $\varphi(1) = \lambda$. Then

- (i) $\sup_{0 < \rho \leq 1} \varphi'(\rho) < \infty$;
- (ii) the limit $\lim_{\rho \rightarrow 0+} T(\rho)$ exists and is finite.

Proof If φ is regular, then (i) and (ii) follow from Theorems 4.2 and 4.1.

Next, we assume that $\varphi(0) > 0$. Using Corollary 4.1, (A 7), (A 8), and (4.3) we obtain

$$|T'(\rho)| \leq c\tau^{1-n}[1 + \tau^\alpha + (\alpha_1\tau)^\alpha] \leq c\tau^{1+\alpha-n} \leq \frac{c}{\rho^{1+\alpha-n}} \quad \text{for } \rho \in (0, 1],$$

which yields

$$|T(\rho)| \leq |T(1)| + c \int_\rho^1 \frac{1}{\rho^{1+\alpha-n}} d\rho \leq |T(1)| + c \quad \text{for } \rho \in (0, 1] \quad (4.39)$$

since $1 + \alpha - n < 1$.

To prove (i) we suppose for a contradiction that there is a sequence $\{\rho_j\} \subset (0, 1]$ such that $\rho_j \rightarrow 0$ and

$$j \leq \varphi'(\rho_j) \leq \alpha_1(\rho_j, \varphi(\rho_j))\tau(\rho_j) \quad \text{for } j \geq 1,$$

where the right hand side of the above inequalities is from Corollary 4.1. It follows from (A 6) and (A 7) that

$$\tau^{1-n}(\rho_j)|\phi'(\varphi'(\rho_j))| \leq c\tau^{1-n}(\rho_j)[1 + \varphi'^{\alpha-1}(\rho_j)] \leq c[\tau^{1-n}(\rho_j) + \tau^{\alpha-n}(\rho_j)],$$

and then $\lim_{j \rightarrow \infty} \tau^{1-n}(\rho_j)\phi'(\varphi'(\rho_j)) = 0$ is true. By (A 2) we have

$$\lim_{j \rightarrow \infty} T(\rho_j) = \lim_{j \rightarrow \infty} h'(\varphi'(\rho_j)\tau^{n-1}(\rho_j)) = +\infty,$$

contradicting with (4.39).

By (i) and $\varphi(0) > 0$ there is some $\rho_0 \in (0, 1]$ such that

$$\varphi'(\rho) < \tau(\rho) \quad \text{for } \rho \in (0, \rho_0].$$

Then $T'(\rho) \geq 0$ for $\rho \in (0, \rho_0]$ by (4.3), which implies that (ii) is true. \square

Cavitating Equilibrium Solutions An equilibrium solution $\varphi \in C^1(0, 1]$ to problem (4.11) is said to be *cavitating* if

- (i) $\varphi(0) > 0$, and
- (ii) $\lim_{\rho \rightarrow 0+} T(\rho) = 0$.

Ball [2] has shown that when $\lambda > 0$ is small there is no cavitating equilibrium solution in the case of the Euclidean space. We present some similar results.

Proposition 4.3 *Let $\mu_+(\max\{\lambda, 1\}) \leq 1$. Let (A 1), (A 2), (A 4), (A 5), (A 6), and (A 7) hold. Suppose that (A 8) holds with $\beta = 0$. If $\varphi \in C^1(0, 1]$ is a cavitating equilibrium solution to problem (4.11), then*

$$\lim_{\rho \rightarrow 0+} \varphi'\tau^{n-1} = \varpi, \quad (4.40)$$

where $\varpi > 0$ is given by $h'(\varpi) = 0$.

Proof By Corollary 4.1, (A 7), and (A 8) with $\beta = 0$, we have $\lim_{\rho \rightarrow 0+} \tau^{n-1} \phi'(\varphi') = 0$. Then $0 = \lim_{\rho \rightarrow 0+} T(\rho) = \lim_{\rho \rightarrow 0+} h'(\varphi' \tau^{n-1})$. (4.40) follows from (A 1). \square

Theorem 4.3 *Let $\mu_+(1) \leq 1$ and let there be $\varepsilon \in (0, 1]$ such that*

$$\kappa(s) = 0 \quad \text{for } s \in (0, \varepsilon). \quad (4.41)$$

Let (A 1), (A 2) and (A 4) hold. Then there is no cavitating equilibrium solution when $\lambda > 0$ is small.

Proof It follows from (4.41) that

$$f'(\rho) = 1, \quad f(\rho) = \rho \quad \text{for } \rho \in [0, \varepsilon]. \quad (4.42)$$

Then the left hand side of the equation in (4.11) equals

$$(n-1)[\phi'(\tau) - \phi'(\varphi') - (\varphi' - \tau)h''(\varphi' \tau^{n-1})\varphi' \tau^{2n-3}] \quad (4.43)$$

for $\rho \in (0, \varepsilon]$ when $\varphi(1) = \lambda < \varepsilon$, where

$$\tau(\rho) = \frac{\varphi(\rho)}{\rho} \quad \text{for } \rho \in (0, \varepsilon].$$

By (A 2) and (A 4), without loss of generality we assume that $\lambda_0 > 0$ is small such that

$$\Phi_1\left(\frac{\lambda_0}{\varepsilon}, \dots, \frac{\lambda_0}{\varepsilon}\right) = \phi'\left(\frac{\lambda_0}{\varepsilon}\right) + h'\left(\left(\frac{\lambda_0}{\varepsilon}\right)^n\right) < 0. \quad (4.44)$$

Let $\varphi \in C^1(0, 1]$ be a cavitating equilibrium solution with $\varphi(1) = \lambda < \lambda_0$ and we derive a contradiction below. We claim

$$\varphi'(\rho) < \tau(\rho) \quad \text{for all } \rho \in (0, \varepsilon]. \quad (4.45)$$

In fact, if there were some $\rho_0 \in (0, \varepsilon]$ such that $\varphi'(\rho_0) = \tau(\rho_0)$, it is easy to check by (4.43) and the uniqueness theorem that

$$\varphi(\rho) = \varphi'(\rho_0)\rho \quad \text{for all } \rho \in (0, \rho_0],$$

which contradicts $\varphi(0) > 0$. Next, by (4.45), (4.3) and (4.44), we obtain

$$\begin{aligned} T(\rho) &< T(\varepsilon) = \tau^{1-n}(\varepsilon)\Phi_1(\varphi'(\varepsilon), \tau(\varepsilon), \dots, \tau(\varepsilon)) \\ &< \tau^{1-n}(\varepsilon)\Phi_1(\tau(\varepsilon), \tau(\varepsilon), \dots, \tau(\varepsilon)) < 0 \quad \text{for } \rho \in (0, \varepsilon), \end{aligned}$$

that contradicts $T(0) = 0$. \square

Theorem 4.4 *Let $\mu_+(1) \leq 1$ and let there be $\varepsilon \in (0, 1]$ such that*

$$\kappa(s) < 0 \quad \text{for } s \in (0, \varepsilon). \quad (4.46)$$

Let (A1), (A2), (A4), (A5), (A6), (A7), and (A8) hold. Further suppose that

- (i) $\phi'(1) + h'(1) = 0$;
- (ii) $\phi'(s) - h''(s^n)s^{2n-1} < 0$ for $s > 1$.

Then there is no cavitating equilibrium solution when $0 < \lambda < \varepsilon$.

Proof We suppose for a contradiction that $\varphi \in C^1(0, 1]$ is a cavitating equilibrium solution with $\varphi(1) = \lambda < \varepsilon$. By Proposition 4.2 there is the largest number $\rho_0 \in (0, 1]$ such that

$$\varphi'(\rho) < \tau(\rho) \quad \text{for all } \rho \in (0, \rho_0). \quad (4.47)$$

Let $\rho_0 \geq \varepsilon$. Then

$$\tau(\rho_0) \leq \frac{f(\lambda)}{f(\varepsilon)} < 1,$$

and therefore there is some $\rho_1 \in (0, \rho_0)$ such that $\tau(\rho_1) = 1$. The relations (4.3) and (4.47) implies that

$$T(\rho) \leq T(\rho_1) = \tau^{1-n}(\rho_1)\Phi_1(\varphi'(\rho_1), 1, \dots, 1) < \tau^{1-n}(\rho_1)\Phi_1(1, \dots, 1) = 0 \quad (4.48)$$

for all $\rho \in (0, \rho_1)$, which contradicts with $T(0) = 0$.

Let $0 < \rho_0 < \varepsilon$. By (4.47) we must have

$$\varphi'(\rho_0) = \tau(\rho_0). \quad (4.49)$$

If $\tau(\rho_0) \leq 1$, we get a contradiction as in (4.48). We assume that $\tau(\rho_0) > 1$. Then

$$\rho_0 < \varphi(\rho_0) < \lambda < \varepsilon. \quad (4.50)$$

By (4.50) and (4.46), we obtain

$$f' \circ \varphi(\rho_0) = f'(\rho_0) - \int_{\rho_0}^{\varphi(\rho_0)} \kappa(s)f(s)ds > f'(\rho_0). \quad (4.51)$$

Thus, it follows from (4.11), (4.49), (4.51) and (ii) that

$$\begin{aligned} & \text{the left hand side of the equation (4.11)} \\ & = (n-1)(f' \circ \varphi - f')[\phi'(\tau) - h''(\tau^n)\tau^{2(n-1)}] < 0 \quad \text{at } \rho = \rho_0, \end{aligned}$$

that is,

$$\varphi''(\rho_0) < 0. \quad (4.52)$$

In addition, we have, by (4.51) and (4.49),

$$f(\rho_0)\tau'(\rho_0) = [f'(\varphi(\rho_0) - f(\rho_0)]\tau(\rho_0) > 0. \quad (4.53)$$

Thus, by (4.52) and (4.53), we obtain

$$\varphi'(\rho) > \varphi'(\rho_0) = \tau(\rho_0) > \tau(\rho) \quad \text{for } \rho < \rho_0 \quad \text{and near } \rho_0,$$

contradicting the definition of ρ_0 . \square

Remark 4.4 *If $\kappa(o) < 0$, assumption (4.46) is true.*

Remark 4.5 *Assumption (i) means that $\Phi_i(1, \dots, 1) = 0$ for $1 \leq i \leq n$ so that the undeformed configuration is a natural state. In addition, if*

$$\phi(v) = \frac{1}{v^\beta} + v^\alpha, \quad h(v) = \frac{2\alpha}{v} + \frac{\alpha + \beta}{2}v^2, \quad \beta > 0, \quad \alpha > 1,$$

then (i) and (ii) hold.

4.3 Energy Minimizers

We seek to minimize

$$I(\varphi) = \int_0^1 f^{n-1}(\rho)\Phi(\rho)d\rho \quad (4.54)$$

among radial deformations φ such that $\varphi(0) \geq 0$ and $\varphi(\rho)$ is increasing. For $\lambda > 0$ given, let

$$\mathcal{A}_\lambda = \left\{ \varphi \in W^{1,1}(0,1) \mid \varphi(0) \geq 0, \varphi' > 0 \text{ a.e., and } I(\varphi) < \infty \right\}.$$

Consider the minimization problem

$$\inf_{\varphi \in \mathcal{A}_\lambda} I(\varphi). \quad (4.55)$$

Let

$$u(p) = \sigma \circ \varphi(\rho),$$

where σ is given by (2.59) and $p = \sigma(\rho)$. Then

$$\varphi(\rho) = \sigma^{-1} \circ u(p), \quad \rho = \sigma^{-1}(p),$$

$$\varphi'(\rho) = \frac{u'(p)}{\sigma' \circ \varphi(\rho)} p'(\rho) = \frac{f^{n-1} \circ \sigma^{-1}(p)}{f^{n-1} \circ \sigma^{-1}(u)} u'(p), \quad \tau(\rho) = \frac{f \circ \sigma^{-1}(u)}{f \circ \sigma^{-1}(p)},$$

and

$$u'(p) = \sigma' \circ \varphi(\rho) \varphi'(\rho) \frac{\partial \rho}{\partial p} = \varphi'(\rho) \tau^{n-1}(\rho).$$

Let

$$\wp(p, u, q) = \phi\left(\frac{f^{n-1} \circ \sigma^{-1}(p)}{f^{n-1} \circ \sigma^{-1}(u)} q\right) + (n-1)\phi\left(\frac{f \circ \sigma^{-1}(u)}{f \circ \sigma^{-1}(p)}\right) + h(q),$$

$$J(u) = \int_0^{\sigma(1)} \wp(p, u, u') dp.$$

Since

$$I(\varphi) = J(u),$$

the minimization problem (4.55) is now equivalent to minimizing $J(u)$ on the set

$$\left\{ u \in W^{1,1}(0, \sigma(1)) \mid u(\sigma(1)) = \sigma(\lambda), u(0) \geq 0, u'(p) > 0 \text{ a.e., and } J(u) < \infty \right\}.$$

A similar argument as in [2] yields

Theorem 4.5 *Suppose that (A1), (A2), (A4), and (A11) hold. I attains an absolute minimum on \mathcal{A}_λ .*

Following the proof of Theorem 7.3 in [2], we obtain the following.

Theorem 4.6 *Let (A1), (A2), (A4), (A7), and (A9) hold. Let φ be a minimizer of I on \mathcal{A}_λ . Then $\varphi \in C^1(0, 1]$, $\varphi'(\rho) > 0$ for all $\rho \in (0, 1]$, $f^{n-1}\Phi_1 \in C^1(0, 1]$ and (4.11) holds for all $\rho \in (0, 1]$. If $\varphi(0) > 0$, then $f^{n-2}\Phi_2 \in L^1(0, 1)$ and*

$$\lim_{\rho \rightarrow 0+} T(\rho) = 0, \quad (4.56)$$

where T is the radial component of the Cauchy stress, given by (4.2).

Next, we have the following.

Theorem 4.7 *Let $\mu_+(\max\{\lambda, 1\}) \leq 1$. Let (A1), (A2), (A4), (A5), (A6), (A7) and (A8) hold. Let $\varphi \in C^1(0, 1]$ be an equilibrium solution to problem (4.11) with $\varphi(1) = \lambda$. Then*

$$I(\varphi) < \infty.$$

Proof By using (4.8) and (4.3), we have

$$\begin{aligned} \varphi' \Phi_1 + (n-1)\tau \Phi_2 &= (n-1)[\tau \phi'(\tau) - \varphi' \phi'(\varphi')] + n\varphi' \tau^{n-1} T \\ &= \frac{f \circ \varphi}{f' \circ \varphi} \tau^{n-1} T' + n\varphi' \tau^{n-1} T \quad \text{for } \rho \in (0, 1). \end{aligned}$$

Thus,

$$f^{n-1}[\varphi' \Phi_1 + (n-1)\tau \Phi_2] = \frac{1}{f' \circ \varphi} (f^n \circ \varphi T)' \quad \text{for } \rho \in (0, 1]. \quad (4.57)$$

Now, using (4.7) and (4.57), we obtain, by integration by parts,

$$\begin{aligned} n \int_\rho^1 f' f^{n-1} \Phi ds + f^n(\rho) \Phi(\rho) &= \left\{ f^n[\Phi - (\varphi' - \tau) \Phi_1] + \frac{f' - f' \circ \varphi}{f' \circ \varphi} f^n \circ \varphi T \right\} \Big|_{\rho=1} \\ &+ \frac{f' \circ \varphi(\rho) - f'(\rho)}{f' \circ \varphi(\rho)} f^n \circ \varphi(\rho) T(\rho) + f^n(\rho) [\varphi'(\rho) - \tau(\rho)] \Phi_1(\rho) \\ &+ \int_\rho^1 \frac{\kappa \circ \varphi \varphi' f' - \kappa f f' \circ \varphi}{f'^2 \circ \varphi} f^n \circ \varphi T ds \quad \text{for } \rho \in (0, 1]. \end{aligned} \quad (4.58)$$

Next, by Proposition 4.2, the term

$$f^n(\rho)[\varphi'(\rho) - \tau(\rho)]\Phi_1(\rho) = f^{n-1} \circ \varphi(\rho)[f(\rho)\varphi'(\rho) - f \circ \varphi(\rho)]T(\rho)$$

converges as ρ goes to 0+ and the limit is finite. Moreover, by Proposition 2.4,

$$\mu_0(1) \leq f'(\rho) \leq e^{\mu_1(1)}, \quad \mu_0(\lambda) \leq f' \circ \varphi(\rho) \leq e^{\mu_1(\lambda)} \quad \text{for } \rho \in (0, 1].$$

Thus, the second term and the last term in the right hand side of (4.58) converge as ρ goes to 0+ and their limits are finite by Proposition 4.2. The proof is complete. \square

4.4 Cavitation

We now derive a cavitating theorem.

Theorem 4.8 *Let $\mu_+(\infty) \leq 1$ and $\mu_-(\infty) < \infty$. Suppose that (A1)-(A9) hold. For any λ sufficiently large a minimizer of I on \mathcal{A}_λ is a cavitating equilibrium solution.*

Proof By Theorem 4.6 a minimizer of I on \mathcal{A}_λ is an equilibrium solution. We suppose for a contradiction that $\varphi \in C^1(0, 1]$ is a regular equilibrium solution with $\varphi(1) = \lambda$ where λ is sufficiently large. For $\varepsilon \in (0, 1)$, let

$$\varphi_\varepsilon(\rho) = \varphi\left\{\sigma^{-1}[(1 - \varepsilon)\sigma(\rho) + \varepsilon\sigma(1)]\right\}.$$

Then

$$\varphi_\varepsilon(1) = \lambda, \quad \varphi_\varepsilon(0) = \varphi(\sigma^{-1}(\varepsilon\sigma(1))) > 0.$$

Thus, $\varphi_\varepsilon \in \mathcal{A}_\lambda$ for all $\varepsilon \in (0, 1)$. We will show that for λ sufficiently large and $\varepsilon \in (0, 1)$ small

$$I(\varphi_\varepsilon) < I(\varphi), \tag{4.59}$$

contradicting that φ is a minimizer of I on \mathcal{A}_λ .

Step 1 Let

$$p = \sigma^{-1}[(1 - \varepsilon)\sigma(\rho) + \varepsilon\sigma(1)], \quad \tau_\varepsilon = \frac{f \circ \varphi_\varepsilon(\rho)}{f(\rho)}, \quad \text{for } \rho \in (0, 1].$$

Simple computations yield

$$p' = (1 - \varepsilon) \frac{f^{n-1}}{f^{n-1} \circ p}, \quad \varphi'_\varepsilon(\rho) = (1 - \varepsilon) \varphi' \circ p(\rho) \frac{f^{n-1}(\rho)}{f^{n-1} \circ p(\rho)}, \tag{4.60}$$

$$\varphi'_\varepsilon(1) = (1 - \varepsilon) \varphi'(1), \quad \tau_\varepsilon = \tau \circ p \frac{f \circ p}{f}, \quad \varphi'_\varepsilon \tau_\varepsilon^{n-1} = (1 - \varepsilon) \varphi' \circ p \tau^{n-1} \circ p. \tag{4.61}$$

In addition, (A 1) gives

$$h(t) \geq h(s) + (t - s)h'(s) \quad \text{for all } t, s \in (0, \infty).$$

By using the formulas above, we obtain

$$\begin{aligned} (1 - \varepsilon) \int_0^1 h(\varphi'_\varepsilon \tau_\varepsilon^{n-1}) f^{n-1} d\rho &= \int_{p(0)}^1 h[(1 - \varepsilon) \varphi' \tau^{n-1}] f^{n-1} d\rho \\ &\leq \int_{p(0)}^1 h(\varphi' \tau^{n-1}) f^{n-1} d\rho - \varepsilon \int_{p(0)}^1 \varphi' \tau^{n-1} h'[(1 - \varepsilon) \varphi' \tau^{n-1}] f^{n-1} d\rho, \end{aligned}$$

which yields

$$\begin{aligned} \int_0^1 h(\varphi' \tau^{n-1}) f^{n-1} d\rho &\geq \varepsilon \int_0^{p(0)} h(\varphi' \tau^{n-1}) f^{n-1} d\rho + (1 - \varepsilon) \int_0^1 h(\varphi'_\varepsilon \tau_\varepsilon^{n-1}) f^{n-1} d\rho \\ &\quad + \varepsilon(1 - \eta) \int_{p(0)}^1 \varphi' \tau^{n-1} h'[(1 - \varepsilon) \varphi' \tau^{n-1}] f^{n-1} d\rho \\ &\quad + \varepsilon \eta \int_{p(0)}^1 \varphi' \tau^{n-1} h'[(1 - \varepsilon) \varphi' \tau^{n-1}] f^{n-1} d\rho, \end{aligned} \quad (4.62)$$

for any $\eta > 0$ small, since $\varepsilon < 1$.

Step 2 By (A 3), there are constants ϖ and $N(\varpi) > 0$ such that

$$\varpi > 1, \quad x h'(x) \geq \varpi h(x), \quad \text{for all } x \geq N(\varpi). \quad (4.63)$$

Since θ is continuous, we take $N(\varepsilon) \geq N(\varpi)$ such that

$$h[(1 - \varepsilon)x] \geq [\theta(1 - \varepsilon) - \varepsilon] h(x) \quad \text{for all } x \geq N(\varepsilon). \quad (4.64)$$

Next, for $\varpi > 1$ being given in (4.63), we fix $\eta > 0$ and $\varepsilon > 0$ small such that

$$\frac{(1 - \eta)\varpi}{1 - \varepsilon} [\theta(1 - \varepsilon) - \varepsilon] \geq 1, \quad (4.65)$$

that is possible because $\lim_{\varepsilon \rightarrow 0} \theta(1 - \varepsilon) = 1$.

By using (4.63)-(4.65), we obtain that, if $\varphi' \tau^{n-1} \geq N(\varepsilon)/(1 - \varepsilon)$ for all $\rho \in [p(0), 1]$, then

$$\begin{aligned} (1 - \eta) \int_{p(0)}^1 \varphi' \tau^{n-1} h'[(1 - \varepsilon) \varphi' \tau^{n-1}] f^{n-1} d\rho &\geq \frac{\varpi}{1 - \varepsilon} \int_{p(0)}^1 h[(1 - \varepsilon) \varphi' \tau^{n-1}] f^{n-1} d\rho \\ &\geq \frac{(1 - \eta)\varpi}{1 - \varepsilon} [\theta(1 - \varepsilon) - \varepsilon] \int_{p(0)}^1 h(\varphi' \tau^{n-1}) f^{n-1} d\rho \geq \int_{p(0)}^1 h(\varphi' \tau^{n-1}) f^{n-1} d\rho. \end{aligned} \quad (4.66)$$

Inserting (4.66) into (4.62) yields

$$\begin{aligned} \int_0^1 h(\varphi' \tau^{n-1}) f^{n-1} d\rho &\geq \int_0^1 h(\varphi'_\varepsilon \tau_\varepsilon^{n-1}) f^{n-1} d\rho \\ &\quad + \frac{\varepsilon \eta}{1 - \varepsilon} \int_{p(0)}^1 \varphi' \tau^{n-1} h'[(1 - \varepsilon) \varphi' \tau^{n-1}] f^{n-1} d\rho, \end{aligned} \quad (4.67)$$

if the following condition holds

$$\varphi' \tau^{n-1} \geq \frac{N(\varepsilon)}{1 - \varepsilon} \quad \text{for all } \rho \in [p(0), 1]. \quad (4.68)$$

Step 3 Since φ is regular, by Corollaries 4.3, 4.4 and Proposition 2.5, we obtain

$$\lambda \frac{\mu_0}{\mu_1} \rho^{c_1-1} \leq \tau(\rho) \leq \lambda \frac{\mu_1}{\mu_0} \rho^{c_0-1}, \quad \lambda \frac{\eta_0 \mu_0}{\mu_1} \rho^{c_1-1} \leq \varphi'(\rho) \leq \lambda \frac{\eta_1 \mu_1}{\mu_0} \rho^{c_0-1}, \quad (4.69)$$

for all $\rho \in (0, 1]$.

By (A 8) and (4.69), we have

$$\phi(\varphi') + (n-1)\phi(\tau) \leq c(p(0))(1 + \lambda^\alpha), \quad \text{for } \lambda \geq 1, \rho \in [p(0), 1], \quad (4.70)$$

where constant $c(p(0)) > 0$ depends on $p(0)$ but is independent of $\lambda \geq 1$. Next, by (4.60), (4.61) and (4.69), we obtain

$$\lambda(1-\varepsilon) \frac{\eta_0 \mu_0}{\mu_1} p^{c_1-1}(0) \frac{f^{n-1}(\rho)}{f^{n-1}(1)} \leq \varphi'_\varepsilon(\rho) \leq \lambda(1-\varepsilon) \frac{\eta_1 \mu_1}{\mu_0} p^{c_0-1}(0) \frac{f^{n-1}(\rho)}{f^{n-1} \circ p(0)} \quad (4.71)$$

for all $\rho \in (0, 1]$, and

$$\lambda p^{c_1-1}(0) \frac{\mu_0}{\mu_1} \frac{f \circ p(0)}{f(\rho)} \leq \tau_\varepsilon(\rho) \leq \lambda \frac{\mu_1}{\mu_0} p^{c_0-1}(0) \frac{f(1)}{f(\rho)} \quad \text{for all } \rho \in (0, 1] \quad (4.72)$$

and

$$\varphi'(\rho) \tau^{n-1}(\rho) \geq \lambda^n \hat{c}(p(0)) \quad \text{for } \rho \in [p(0), 1]. \quad (4.73)$$

It follows from (A 8), (4.71) and (4.72) that

$$\phi(\varphi'_\varepsilon) + (n-1)\phi(\tau_\varepsilon) \leq c(p(0)) \left(1 + \lambda^\alpha \left[1 + \frac{1}{f^\alpha(\rho)} \right] + \frac{1}{f^{\beta(n-1)}(\rho)} + \frac{1}{f^\beta(\rho)} \right) \quad (4.74)$$

for all $\lambda \geq 1$ and $\rho \in (0, 1]$.

By using (A 4), (4.67), (A 8) and (4.71)-(4.74), we obtain

$$\begin{aligned} I(\varphi) &\geq \int_{p(0)}^1 [\phi(\varphi') + (n-1)\phi(\tau)] f^{n-1} d\rho + \int_0^1 h(\varphi' \tau^{n-1}) f^{n-1} d\rho \\ &\geq I(\varphi_\varepsilon) + \int_{p(0)}^1 [\phi(\varphi') + (n-1)\phi(\tau)] f^{n-1} d\rho \\ &\quad - \int_0^1 [\phi(\varphi'_\varepsilon) + (n-1)\phi(\tau_\varepsilon)] f^{n-1} d\rho \\ &\quad + \frac{\varepsilon \eta}{1-\varepsilon} \int_{p(0)}^1 \varphi' \tau^{n-1} h'[(1-\varepsilon)\varphi' \tau^{n-1}] f^{n-1} d\rho \\ &\geq I(\varphi_\varepsilon) + \frac{\varepsilon \eta}{1-\varepsilon} \lambda^n \hat{c}(p(0)) h'[(1-\varepsilon)\lambda^n \hat{c}(p(0))] \int_{p(0)}^1 f^{n-1}(\rho) d\rho \\ &\quad - c(p(0))(1 + \lambda^\alpha), \end{aligned} \quad (4.75)$$

when

$$\lambda \geq \max \left\{ \left(\frac{N(\varepsilon)}{\hat{c}(p(0))(1-\varepsilon)} \right)^{1/n}, 1 \right\}.$$

The proof is complete. \square

5 Cavitation for Membrane Shells

The nonlinear shell membrane energy is obtained in [6] by the Γ -limit of the sequence of three-dimensional energies. Here we will show that such membrane energies may take a form of (2.5) where M is a surface in \mathbb{R}^3 and g is the induced metric of M from \mathbb{R}^3 .

We consider a homogeneous elastic material with stored-energy function $\hat{W} : M_+^{3 \times 3} \rightarrow \mathbb{R}$. Suppose that \hat{M} is frame-indifferent and isotropic, that is,

$$\hat{W}(F) = \hat{W}(QFR) \quad \text{for } F \in M_+^{3 \times 3}, \quad Q, R \in \text{SO}(3). \quad (5.1)$$

We assume that a middle surface S is a bounded, connected open set of a C^2 surface in \mathbb{R}^3 and let N be the normal field of S . For $h > 0$ given, we consider the set Ω_h defined by

$$\Omega_h = \{ x + sN(x) \mid x \in S, |s| < h \}.$$

This set is the reference configuration of a shell with thickness $2h$. Let $\mathbf{u} : \Omega_h \rightarrow \mathbb{R}^3$ be a deformation of the shell. Then the stored energy is

$$E_h(\mathbf{u}) = \int_{\Omega_h} \hat{W}(\nabla \mathbf{u}(x)) dx = \int_{-h}^h \int_S \hat{W}(\nabla \mathbf{u}(x + sN(x))) A(x, s) dg ds,$$

where g is the induced metric on S from \mathbb{R}^3 and

$$A(x, s) = 1 + sH + s^2\kappa,$$

$H/2$ is the mean curvature, and κ is the Gaussian curvature in S .

For a deformation $\mathbf{u} \in L^p(\Omega_1, \mathbb{R}^3)$, we define $\mathbf{u}_h \in L^p(\Omega_h, \mathbb{R}^3)$ by

$$\mathbf{u}_h(x + shN(x)) = \mathbf{u}(x + sN(x)) \quad \text{for } x + shN(x) \in \Omega_h. \quad (5.2)$$

We define the rescaled energies by

$$I_h(\mathbf{u}) = \frac{1}{h} E_h(\mathbf{u}_h) = \int_{-1}^1 \int_S \hat{W}(\nabla \mathbf{u}_h(x + hsN(x))) A(x, sh) dg ds,$$

for $\mathbf{u} \in L^p(\Omega_1, \mathbb{R}^3)$ and $h > 0$ small.

Let $x \in S$ be fixed. Let e_1, e_2 be an orthonormal basis of S_x such that e_1, e_2, e_3 is an orthonormal basis of \mathbb{R}^3 with positive orientation, where $e_3 = N(x)$, to satisfy

$$\hat{D}_{e_i} N = \kappa_i e_i \quad \text{at } x \quad \text{for } i = 1, 2, \quad (5.3)$$

where \hat{D} is the connection of the Euclidean space \mathbb{R}^3 and κ_i are the eigenvalues of the second fundamental form of S at x . Thus, $H(x) = \kappa_1 + \kappa_2$ and $\kappa(x) = \kappa_1 \kappa_2$.

It follows from (5.2) that

$$(1 + sh\kappa_i) \mathbf{u}_{h*} e_i = (1 + s\kappa_i) \mathbf{u}_* e_i, \quad h \mathbf{u}_{h*} e_3 = \mathbf{u}_* e_3. \quad (5.4)$$

By (5.1) and (5.4), we obtain

$$\begin{aligned}\hat{W}(\nabla \mathbf{u}_h(x + hsN(x))) &= \hat{W}(\mathbf{u}_{h*}e_1 \mid \mathbf{u}_{h*}e_2 \mid \mathbf{u}_{h*}e_3) \\ &= \hat{W}\left(\frac{1 + s\kappa_1}{1 + sh\kappa_1}\mathbf{u}_*e_1 \mid \frac{1 + s\kappa_2}{1 + sh\kappa_2}\mathbf{u}_*e_2 \mid \frac{1}{h}\mathbf{u}_*e_3\right).\end{aligned}\quad (5.5)$$

In particular, if we let

$$\varphi^h(x + sN(x)) = \varphi(x) + shw(x) \quad \text{for } \varphi, w \in W^{1,p}(S, \mathbb{R}^3),$$

then

$$\hat{W}(\nabla \varphi_h^h(x + shN)) = \hat{W}\left(\frac{\varphi_*e_1 + shw_*e_1}{1 + sh\kappa_1} \mid \frac{\varphi_*e_2 + shw_*e_2}{1 + sh\kappa_2} \mid w\right).$$

Let

$$\hat{W}_0(F_1, F_2) = \min_{z \in \mathbb{R}^3} \hat{W}(F_1 \mid F_2 \mid z) \quad \text{for } F_1, F_2 \in \mathbb{R}^3.$$

Let $\mathcal{Q}\hat{W}_0 = \sup\{Z : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}, Z \text{ quasiconvex}, Z \leq \hat{W}_0\}$ be the quasiconvex envelope of \hat{W}_0 .

We introduce the space

$$V_M = \{\varphi \in W^{1,p}(\Omega_1, \mathbb{R}^3) \mid \varphi_*N = 0 \text{ for } x \in S\}, \quad (5.6)$$

for which we call the space of membrane displacements.

By similar arguments as in [6], we may have

Theorem 5.1 ([6]) *Let all the assumptions on the function \hat{W} in [6] hold. Further suppose (5.1) is true. Then the sequence I_h Γ -converges for the strong topology of $L^p(\Omega_1, \mathbb{R}^3)$ when $h \rightarrow 0$. For $\varphi \in L^p(\Omega_1, \mathbb{R}^3)$, it's Γ -limit is given by*

$$I_0(\varphi) = \begin{cases} 2 \int_S \mathcal{Q}\hat{W}_0(\varphi_*e_1, \varphi_*e_2) dg & \text{if } \varphi \in V_M, \\ +\infty & \text{otherwise.} \end{cases} \quad (5.7)$$

Next, we shall reformulate the Γ -limit energy (5.7) to relate it to the energy formula (2.5).

Let us introduce a function $W_0 : M_+^{2 \times 2} \rightarrow \mathbb{R}$ by

$$W_0(F) = \min_{z \in \mathbb{R}^3} \hat{W}\begin{pmatrix} F & Z_1 \\ 0 & z_3 \end{pmatrix},$$

where $z = (z_1, z_2, z_3)$ and $Z_1 = (z_1, z_2)^T$. It is easy to check that

$$W_0(QFR) = W_0(F) \quad \text{for } F \in M_+^{2 \times 2}, \quad Q, R \in \text{SO}(2). \quad (5.8)$$

Let $W = \sup\{Z : M^{2 \times 2} \rightarrow \mathbb{R}, Z \text{ quasiconvex}, Z \leq W_0\}$ be the quasiconvex envelope of W_0 . From Dacorogna's representation formula for the quasiconvex envelope of W_0 , we obtain

$$W(F) = \frac{1}{\pi} \inf_{\mathcal{X} \in W_0^{1,\infty}(D, \mathbb{R}^2)} \int_D W_0(F + \nabla \mathcal{X}) dy,$$

where D is the unit disc in \mathbb{R}^2 . The formula above yields, by (5.8),

$$W(QFR) = W(F) \quad \text{for } F \in M_+^{2 \times 2}, \quad Q, R \in \text{SO}(2). \quad (5.9)$$

For $\varphi \in C^1(S, \mathbb{R}^3)$, consider a C^1 surface given by

$$S_\varphi = \{ \varphi(x) \mid x \in S \}.$$

Denote the induced metric of S_φ from \mathbb{R}^3 by $g_\varphi = \langle \cdot, \cdot \rangle \circ \varphi$.

We have the following.

Proposition 5.1 *Let the functional I_0 be given by Theorem 5.1. Then*

$$I_0(\varphi) = 2 \int_S W(d\varphi(x)) dg \quad \text{for } \varphi \in C^1(S, \mathbb{R}^3), \quad (5.10)$$

where

$$d\varphi(x) = \left(\langle E_i, \varphi_* e_j \rangle (\varphi(x)) \right)_{2 \times 2},$$

and, e_1, e_2 and E_1, E_2 are positively orientated orthonormal bases of S_x and $(S_\varphi)_{\varphi(x)}$, respectively.

Proof Let E_3 be the normal field of S_φ . Then $\langle E_3, \varphi_* e_i \rangle = 0$ for $x \in S$ and $i = 1, 2$. Since $\varphi_* e_i = \sum_{j=1}^3 \langle E_j, \varphi_* e_i \rangle (\varphi(x)) E_j$ for $1 \leq j \leq 3$, i.e.,

$$\left(\varphi_* e_1 \mid \varphi_* e_2 \mid z \right) = \left(E_1 \mid E_2 \mid E_3 \right) \begin{pmatrix} \langle E_1, \varphi_* e_1 \rangle & \langle E_1, \varphi_* e_2 \rangle & \langle E_1, z \rangle \\ \langle E_2, \varphi_* e_1 \rangle & \langle E_2, \varphi_* e_2 \rangle & \langle E_2, z \rangle \\ 0 & 0 & \langle E_3, z \rangle \end{pmatrix},$$

for all $z \in \mathbb{R}^3$, we obtain, by (5.1),

$$\hat{W}_0(\varphi_* e_1, \varphi_* e_2) = W_0(d\varphi) \quad \text{for } x \in S.$$

The proof is complete. □

Next, let us assume that $M \subset \mathbb{R}^3$ is a C^2 surface with the induced metric g . Suppose a middle surface of a shell S is a bounded, open set of M . We assume that all deformations of S are confined in M . For such a deformation \mathbf{u} , we assume that the stored energy is given

$$E(\mathbf{u}) = \int_S W(d\mathbf{u}) dg, \quad (5.11)$$

where $W : M_+^{2 \times 2} \rightarrow \mathbb{R}$ satisfies (5.9).

Membrane Shells of Revolution Let ψ be a C^2 function on $[0, \infty)$ with $\psi'(0) = 0$. Consider a surface of revolution given by

$$M = \{ (x, \psi(r)) \in \mathbb{R}^3 \mid x = (x_1, x_2) \in \mathbb{R}^2, \quad r = |x| \}.$$

The Gaussian curvature is

$$\kappa(p) = \frac{\psi'(r)\psi''(r)}{r(1+\psi'^2(r))^2} \quad \text{for } p = (x, \psi(r)) \in M.$$

The normal field is

$$N(p) = \frac{1}{\sqrt{1+\psi'^2(r)}} \left(-\frac{\psi'(r)}{r}x, 1 \right).$$

Let $o = (0, \psi(0)) \in M$ be fixed. Then $M_o = \mathbb{R}^2$. Let $\zeta(t)$ be defined by the equation

$$t = \int_0^{\zeta(t)} \sqrt{1+\psi'^2(s)} ds \quad \text{for } t \geq 0. \quad (5.12)$$

Let

$$\gamma(t) = \left(\zeta(t)v, \psi(\zeta(t)) \right) \quad \text{for } t \in \mathbb{R},$$

where $v = (v_1, v_2) \in \mathbb{R}^2$ with $v_1^2 + v_2^2 = 1$.

Lemma 5.1 $\gamma(t)$ is a normal geodesic such that

$$\gamma(0) = o, \quad \dot{\gamma}(0) = v.$$

Proof Let D denote the connection of the induced metric g of surface M . Then

$$D_{\dot{\gamma}(t)}\dot{\gamma} = \ddot{\gamma}(t) - \langle \ddot{\gamma}(t), N(\gamma(t)) \rangle N(\gamma(t)) = 0 \quad \text{for } t \geq 0,$$

which prove the lemma. \square

It follows from Lemma 5.1 that

$$\kappa(t) = \kappa(\gamma(t)) = \frac{\psi'(\zeta(t))\psi''(\zeta(t))}{\zeta(t)(1+\psi'^2(\zeta(t)))^2} \quad \text{for } t \geq 0, \quad (5.13)$$

where ζ is given by (5.12).

In addition, we have the following.

Proposition 5.2 (M, g, o) is a model.

Proof Since $n = 2$, we have

$$\mathbf{R}(\dot{\gamma}(t), X, \dot{\gamma}(t), X) = \kappa(\gamma(t))|X|^2 \quad \text{for all } X \in M_{\gamma(t)}, \quad \langle X, \dot{\gamma}(t) \rangle = 0,$$

where $\mathbf{R}(\cdot, \cdot, \cdot, \cdot)$ is the curvature tensor, which imply that formula (2.12) holds true. By Proposition 2.3, the proof is complete. \square

Let a middle surface \mathbf{B} of a membrane shell be the unit geodesic disc in (M, g) centered at the point o , i.e.,

$$\mathbf{B} = \left\{ \left(\zeta(t)v, \psi(\zeta(t)) \right) \mid v \in \mathbb{R}^2, 0 \leq t < 1 \right\}.$$

The radial deformations are given by

$$\mathbf{u}(p) = \left(\zeta \circ \varphi(\rho)v, \psi \circ \zeta \circ \varphi(\rho) \right) \quad \text{for } p = \left(\zeta(\rho)v, \psi(\zeta(\rho)) \right),$$

where φ is a function on $[0, \infty)$. Thus, all the theorems, corollaries, and propositions in Sections 2-4 hold true for radial deformations \mathbf{u} . We do not repeat them here.

To end this section, we present two examples which verify the assumptions on the radial curvature in Proposition 3.2 and Theorem 4.8, respectively.

Example 5.1 Let $\varepsilon > 0$ be given and let $\psi_0 \in C_0^\infty(0, \infty)$ be such that

$$\psi_0(t) = 0 \quad \text{for } 0 \leq t \leq \varepsilon; \quad \psi_0(t) = 1 \quad \text{for } t \geq 2\varepsilon.$$

Let

$$\psi_a(t) = \kappa_0 + \frac{a\psi_0(t)}{1+t} \quad \text{for } t \geq 0,$$

where $\kappa_0 < 0$ is a constant and $a > 0$ is such that

$$\int_0^\infty s(\kappa_a)_+(s)ds = \int_\varepsilon^{2\varepsilon} s(\kappa_a)_+(s)ds \leq 1.$$

Consider the incompressible case. Let Φ satisfy the Baker-Ericksen inequalities with $\hat{\Phi}''(1) > 0$ and such that (3.26) is true. Let

$$I(A) = \int_{\mathbf{B}} W(d\mathbf{u})dg - P \int_{\mathbf{S}} \rho(\mathbf{u})d\mathbf{S}.$$

It follows from Proposition 3.2 (i) that, for $A > 0$ small and $P = \chi(A)$, A is a local minimizer of $I(A)$.

Example 5.2 Let

$$\psi(s) = a \log(1 + s^2) \quad \text{for } s \geq 0,$$

where $a > 0$ is a constant. If $0 < a \leq 1/\sqrt{2}$, then

$$\int_0^\infty t\kappa_+(t)dt \leq 1, \quad \int_0^\infty t\kappa_-(t)dt < \infty. \quad (5.14)$$

By (5.13), we have

$$\kappa(t) = 4a^2 \frac{1 - \zeta^4(t)}{[1 + (4a^2 + 2)\zeta^2(t) + \zeta^4(t)]^2} \quad \text{for } t \geq 0.$$

Let $t_0 > 0$ be given by $\zeta(t_0) = 1$. Thus,

$$\begin{aligned} \int_0^\infty t\kappa_+(t)dt &= 4a^2 \int_0^{t_0} \frac{t[1 - \zeta^4(t)]}{[1 + (4a^2 + 2)\zeta^2(t) + \zeta^4(t)]^2} dt \\ &\leq 4a^2 \int_0^1 \frac{1 - \zeta^2}{1 + (4a^2 + 2)\zeta^2 + \zeta^4} d\zeta \leq 4a^2 \int_0^1 \frac{1 - \zeta^2}{(1 + \zeta^2)^2} d\zeta = 2a^2. \end{aligned}$$

Similar arguments yield the second estimate in (5.14).

Consider the compressible case. Let W be given by (4.8) with $n = 2$ such that (A1)-(A9) hold true. It follows from Theorem 4.8 that, for any λ sufficiently large, a minimizer of I on \mathcal{A}_λ is a cavitating equilibrium solution, where I is given by (4.54).

6 Cavitation for Ellipsoids in \mathbb{R}^n

Let

$$g(x) = G(x)$$

be a symmetric, C^2 , and positively definite matrix for each $x \in \mathbb{R}^n$ and regard the pair (\mathbb{R}^n, g) as a Riemannian manifold. The study in the previous sections describes radial deformations of a ball-like body if we apply it to the Riemannian manifold (\mathbb{R}^n, g) . The existence of corresponding cavitating equilibrium solutions depends on the geometric properties of the metric g and on the growth properties of the constitutive function W together.

We denote the metric g on \mathbb{R}^n by

$$g(X, Y) = \langle X, Y \rangle_g = \langle G(x)X, Y \rangle \quad \text{for } X, Y \in \mathbb{R}_x^n = \mathbb{R}^n,$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean metric in \mathbb{R}^n . Then under the natural coordinates $x = (x_1, \dots, x_n)$

$$g_{ij}(x) = \left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\rangle_g = \left\langle G(x) \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\rangle \quad \text{for } x \in \mathbb{R}^n.$$

Consider a body which occupies the open subset Ω of \mathbb{R}^n . A map $\mathbf{u} : \Omega \rightarrow \mathbb{R}^n$ is said to be a deformation of the body Ω .

Theorem 6.1 *Let $W : M_+^{n \times n} \rightarrow \mathbb{R}$ be a constitutive function and satisfy (2.2). Then*

$$W(d\mathbf{u}) = W(G^{1/2}(\mathbf{u}(x))\nabla\mathbf{u}(x)G^{-1/2}(x)), \quad (6.1)$$

where $\mathbf{u} = (u_1, \dots, u_n)$ is a deformation, $W(d\mathbf{u})$ is defined by (2.3) on the Riemannian manifold (\mathbb{R}^n, g) and

$$\nabla u = \left(\frac{\partial u_i}{\partial x_j}(x) \right) \quad \text{for } x \in \mathbb{R}^n$$

is the gradient matrix of the map \mathbf{u} in the Euclidean space \mathbb{R}^n .

Proof Let $x \in \mathbb{R}^n$ be given. Let $\{e_i\}$ and $\{E_i\}$ be orthonormal bases of $(\mathbb{R}_x^n, g(x))$ and $(\mathbb{R}_{\mathbf{u}(x)}^n, g \circ \mathbf{u}(x))$ with positive orientation, respectively.

Let

$$e_i = \sum_{j=1}^n \alpha_{ij} \partial_{x_j}|_x, \quad E_i = \sum_{j=1}^n \beta_{ij} \partial_{x_j}|_{\mathbf{u}(x)}.$$

Then the relations,

$$\delta_{ij} = \sum_{k=1}^n \alpha_{ik} \langle \partial_{x_k}, e_j \rangle_g, \quad \text{for } 1 \leq i, k \leq n,$$

imply that

$$\left(\alpha_{ij} \right) \left(\langle \partial_{x_i}, e_j \rangle_g \right) = I. \quad (6.2)$$

On the other hand, the relations,

$$\langle e_i, \partial_{x_j} \rangle_g = \sum_{k=1}^n \alpha_{ik} \langle \partial_{x_k}, \partial_{x_j} \rangle_g = \sum_{k=1}^n \alpha_{ik} g_{kj}(x), \quad \text{for } 1 \leq i, j \leq n,$$

yield

$$\left(\langle e_i, \partial_{x_j} \rangle_g \right) = \left(\alpha_{ij} \right) G(x), \quad (6.3)$$

where $G(x) = (g_{ij}(x))$.

It follows from formulas (6.2) and (6.3) that

$$\left(\alpha_{ij} \right) G(x) \left(\alpha_{ij} \right)^T = I,$$

where the superscript "T" denotes the transpose. A similar computation gives

$$\left(\beta_{ij} \right) G(\mathbf{u}(x)) \left(\beta_{ij} \right)^T = I.$$

Noting the relations

$$\mathbf{u}_* \partial_{x_i} = \sum_{k=1}^n u_{kx_i} \partial_{x_k} |_{\mathbf{u}(x)},$$

we obtain

$$d\mathbf{u}(E_i, e_j) = \langle E_i, \mathbf{u}_* e_j \rangle_g \circ \mathbf{u}(x) = \sum_{klp} \beta_{ik} g_{kp}(\mathbf{u}(x)) u_{px_l} \alpha_{jl},$$

that is,

$$\left(d\mathbf{u}(E_i, e_j) \right) = \left(\beta_{ij} \right) G(\mathbf{u}(x)) \nabla \mathbf{u}(x) \left(\alpha_{ij} \right)^T = Q G^{1/2}(\mathbf{u}(x)) \nabla u(x) G^{-1/2}(x) R,$$

where

$$Q = \left(\beta_{ij} \right) G^{1/2}(\mathbf{u}(x)), \quad R = G^{1/2}(x) \left(\alpha_{ij} \right)^T,$$

belong to $\text{SO}(n)$. Thus, formula (6.1) follows from assumption (2.2). \square

Total Stored Energy Let dg denote the volume element of \mathbb{R}^n in the metric g . Then

$$dg = \det^{1/2} G(x) dx \quad \text{for } x \in \mathbb{R}^n$$

where dx is the volume element of \mathbb{R}^n in the Euclidean metric. Let $W : M_+^{n \times n} \rightarrow \mathbb{R}$ be a constitutive function and satisfy (2.2). By Theorem 6.1, in a typical deformation in which the particle $x \in \Omega$ is displaced to $\mathbf{u}(x) \in \mathbb{R}^n$ energy (2.5) becomes

$$E(\mathbf{u}) = \int_{\Omega} \hat{W}(x, \mathbf{u}(x), \nabla \mathbf{u}(x)) dx, \quad (6.4)$$

where

$$\hat{W}(x, y, F) = W(G^{1/2}(y) F G^{-1/2}(x)) \det^{1/2} G(x), \quad (6.5)$$

for $x, y \in \mathbb{R}^n$ and $F \in M_+^{n \times n}$, is the total constitutive law. Then

$$\hat{W}(x, y, F) = \Phi\left(v_1(x, y, F), \dots, v_n(x, y, F)\right)^{1/2} \det G(x),$$

where $v_1(x, y, F), \dots, v_n(x, y, F)$ denote the singular vaules of $G^{1/2}(x)FG^{-1/2}(y)$ for $x, y \in \mathbb{R}^n$ and $F \in M_+^{n \times n}$. Thus, formula (6.4) is composed by the constitutive function W and the matrices $G(x)$ together.

Example 6.1 *Introduce a metric on \mathbb{R}^n by*

$$g = e^{2a(r)}I \quad \text{for } r = |x| \in \mathbb{R}^n, \quad (6.6)$$

where $a(s)$ is a C^2 function on $[0, \infty)$ and I is the unit matrix. Suppose a constitutive function $W = \Phi$ is given by (4.8). Then the energy ensity (6.5) is

$$\hat{W}(x, y, F) = W\left(q(x, y)F\right)e^{a(|x|)} = \sum_{i=1}^n \phi\left(q(x, y)v_i\right)e^{na(|x|)} + h\left(q^n(x, y)v_1 \cdots v_n\right)e^{na(|x|)}$$

for $x, y \in \mathbb{R}^n$, where

$$q(x, y) = \frac{e^{a(|y|)}}{e^{a(|x|)}}$$

and v_1, \dots, v_n denote the singular vaules of F for $F \in M_+^{n \times n}$.

We take o to be the origin 0 in \mathbb{R}^n to consider what conditions on g are needed for (\mathbb{R}^n, g, o) being a model.

Proposition 6.1 *Let $G(x)$ satisfy*

$$G(x)x = b(\eta(x))Ax \quad \text{for } x \in \mathbb{R}^n, \quad (6.7)$$

where $\eta(x) = \sqrt{\langle Ax, x \rangle}$, A is a sysmetric, positive, and constant matrix, and b is a positive C^2 function on $[0, \infty)$ with $b(0) = 1$. Then

- (i) (\mathbb{R}^n, g, o) is a model;
- (ii) A geodesic ball of (\mathbb{R}^n, g) centered at o is an ellipsoids of the Euclidean space \mathbb{R}^n .

Proof (i) We introduce one more metric g_1 on \mathbb{R}^n by

$$g_A(X, Y) = \langle AX, Y \rangle \quad \text{for } X, Y \in \mathbb{R}_x^n, x \in \mathbb{R}^n.$$

Clearly, (6.7) implies $G(0) = g_A$. We now have three metrics on \mathbb{R}^n , that are g , g_A , and the Euclidean metric $\langle \cdot, \cdot \rangle$.

Since A is constant, for $x \in \mathbb{R}^n$ given, $x \neq 0$, the curve

$$\alpha(t) = t \frac{x}{\eta(x)} \quad \text{for } t \geq 0,$$

is a normal geodesic in (\mathbb{R}^n, g_A) initiating at o . Thus, $\eta(x)$ is the distance function of (\mathbb{R}^n, g_A) from $x \in \mathbb{R}^n$ to o and

$$\nabla_A \eta = \frac{x}{\eta} \quad \text{for } x \in \mathbb{R}^n, x \neq 0,$$

where ∇_A denotes the Levi-Civita connection of (\mathbb{R}^n, g_A) .

Let σ be the solution to problem

$$\sigma'(t) = b^{-1/2}(\sigma(t)) \quad \text{for } t > 0, \quad \sigma(0) = 0.$$

Set

$$\gamma(t) = \sigma(t)X(x) \quad \text{for } t > 0, \tag{6.8}$$

where $X(x) = \nabla_A \eta$. Next, we prove that $\gamma(t)$ is a normal geodesic of (\mathbb{R}^n, g) .

Let D be the Levi-Civita connection of (\mathbb{R}^n, g) . We compute $D_X X$. Let $x_0 \in \mathbb{R}^n$ be given. Let Z be a constant vector satisfying $\langle x_0, Z \rangle_{g(x_0)} = 0$. Then

$$\langle (\nabla_A \eta)(x_0), Z \rangle_A = \frac{1}{\eta} \langle Ax_0, Z \rangle = \frac{1}{\eta(x_0)b(\eta(x_0))} \langle G(x_0)x_0, Z \rangle = 0,$$

$$[X(x_0), Z] = (\nabla_A)_{X(x_0)} Z - (\nabla_A)_Z \nabla_A \eta = -\frac{1}{\eta(x_0)} Z \quad (\text{since } A \text{ is constant}),$$

$$\langle X, Z \rangle_g = \langle G(x) \frac{x}{\eta}, Z \rangle = b(\eta) \langle \nabla_A \eta, Z \rangle_A.$$

We have

$$\begin{aligned} \langle D_X X, Z \rangle_{g(x_0)} &= X \langle X, Z \rangle_g - \langle X, D_X Z \rangle_g = X(b) \langle \nabla_A \eta(x_0), Z \rangle_A - \langle X, D_Z X \rangle_g - \langle X, [X, Z] \rangle_g \\ &= -\frac{1}{2} Z |X|_g^2 = -\frac{1}{2} Z (b |\nabla_A \eta|_A^2) = -\frac{b'(\eta)}{2} \langle \nabla_A \eta, Z \rangle_A = 0, \end{aligned}$$

for all $Z \in \mathbb{R}^n$ satisfying $\langle x_0, Z \rangle_{g(x_0)} = 0$. Thus, we obtain

$$D_{X(x_0)} X = \langle D_X X, \frac{x_0}{|x_0|_{g(x_0)}} \rangle_{g(x_0)} \frac{x_0}{|x_0|_{g(x_0)}} = \frac{b'(\eta)}{2b(\eta)} X(x_0).$$

It follows that

$$D_{\dot{\gamma}_0(t)} \dot{\gamma}_0 = \ddot{\sigma}(t) X + \dot{\sigma}^2(t) D_{X(x_0)} X = [\ddot{\sigma}(t) + \frac{b'(\eta)}{2b(\eta)}] X(x_0) = 0,$$

where $\gamma_0(t) = \sigma(t) \frac{x_0}{\eta(x_0)}$.

Let ρ be the distance function from $x \in \mathbb{R}^n$ to o in the metric g . By (6.8), we obtain

$$\rho(x) = \sigma^{-1}(\eta(x)) \quad \text{for } x \in \mathbb{R}^n,$$

and

$$\exp_o t \frac{x}{\eta(x)} = \sigma(t) \frac{x}{\eta(x)} \quad \text{for } t \geq 0, x \in \mathbb{R}^n. \tag{6.9}$$

Let $\psi : (\mathbb{R}^n, g_A) \rightarrow (\mathbb{R}^n, g_A)$ be a linear isometry. Then the operator Ψ of (2.11) is given by

$$\Psi(x) = \psi x \quad \text{for } x \in \mathbb{R}^n,$$

which imply that $\Psi : \exp_o \Sigma(o) \rightarrow \exp_o \Sigma(o)$ is an isometry.

(ii) Let $\mathbf{B}(t)$ be the geodesic ball centered at o with radius $t > 0$. It follows from (6.9) that

$$\begin{aligned} \mathbf{B}(t) &= \left\{ \sigma(s) \frac{x}{\eta(x)} \mid x \in \mathbb{R}^n, 0 \leq s < t \right\} \\ &= \left\{ y \mid y \in \mathbb{R}^n, \sqrt{\langle Ay, y \rangle} < \sigma(t), 0 \leq s < t \right\}, \end{aligned} \quad (6.10)$$

that are ellipsoids in \mathbb{R}^n . □

Remark 6.1 If $A = \text{diag} \{ \frac{1}{a_1^2}, \dots, \frac{1}{a_n^2} \}$, then the geodesic balls (6.10) are

$$\mathbf{B}(t) = \left\{ y \mid y \in \mathbb{R}^n, \sum_{i=1}^n \frac{y_i^2}{a_i^2} < \sigma^2(t) \right\}$$

and the geodesic spheres are

$$\left\{ y \mid y \in \mathbb{R}^n, \sum_{i=1}^n \frac{y_i^2}{a_i^2} = \sigma^2(t) \right\} \quad \text{for } t > 0.$$

Remark 6.2 Let

$$\Gamma_{ij}(x) = x_i \partial x_j - x_j \partial x_i \quad x \in \mathbb{R}^n, \quad 1 \leq i, j \leq n.$$

Let A be a constant matrix. Then matrices

$$G(x) = b_1(\eta)A + b_2(\eta)Ax \otimes Ax + \sum_{ij} b_{ij}(x)\Gamma_{ij}(x) \otimes \Gamma_{ij}(x) \quad \text{for } x \in \mathbb{R}^n$$

meet conditions (6.7) since

$$G(x)x = [b_1(\eta) + b_2(\eta)\eta^2]Ax \quad \text{for } x \in \mathbb{R}^n.$$

Let $G(x)$ satisfy assumptions (6.7). Then radial deformations are given by

$$\mathbf{u}(x) = \sigma \circ \varphi(\rho) \frac{x}{\eta(x)} \quad \rho = \sigma^{-1}(\eta(x)), \quad x \in \mathbb{R}^n.$$

Thus, all the theorems, corollaries, and propositions in Sections 2-4 hold true for radial deformations above if we apply them to the model (\mathbb{R}^n, g, o) .

Finally, let us consider some situations for which the radial curvature assumptions in Theorem 4.8 hold. Let κ be a C^1 function in $[0, \infty)$ such that

$$\int_0^\infty s\kappa_+(s)ds \leq 1, \quad \int_0^\infty s\kappa_-(s)ds < \infty,$$

where $\kappa_+ = \max\{0, \kappa\}$ and $\kappa_- = \min\{0, -\kappa\}$. Suppose f is the solution to problem

$$f''(t) + \kappa(t)f(t) = 0 \quad \text{for } t > 0; \quad f(0) = 0, \quad f'(0) = 1.$$

By similar arguments as in the proof of Proposition 4.2 in [4], we obtain the following.

Proposition 6.2 *Let A be a symmetric, positive, and constant matrix. Let*

$$G(x) = \frac{1}{f^2(\eta)}A + \frac{1}{\eta^2}\left[1 - \frac{f^2(\eta)}{\eta^2}\right]Ax \otimes Ax \quad \text{for } x \in \mathbb{R}^n,$$

where $\eta = \sqrt{\langle Ax, x \rangle}$. Then

- (i) $G(x)$ are symmetric and positive for all $x \in \mathbb{R}^n$.
- (ii) (\mathbb{R}^n, g, o) is a model.
- (iii) The radial curvature is $\kappa(t)$ for $t \geq 0$.

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